

Extending positive definiteness

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To the memory of M.G. Kreĭn (1907-1989)

ABSTRACT. The main result of the paper gives criteria for extendibility of sesquilinear form-valued mappings defined on symmetric subsets of $*$ -semigroups to positive definite ones. By specifying this we obtain new solutions of:

- the truncated complex moment problem,
- the truncated multidimensional trigonometric moment problem,
- the truncated two-sided complex moment problem,

as well as characterizations of unbounded subnormality and criteria for the existence of unitary power dilation.

Introduction. In [63] a fairly general concept of $*$ -semigroups, which includes groups, $*$ -algebras and quite a number of instances in between, and positive definite functions on them has been originated by Sz.-Nagy. On the other hand, there are different notions which are related to positive definiteness: positivity of sequences in the theory of moments and complete positivity in C^* -algebras. Positivity understood in the sense of Marcel Riesz and Haviland¹ usually ensures the sequence to be a moment one while complete positivity works for dilations (Stinespring, and what is equivalent, Sz.-Nagy, see [60] for the argument).

If one goes beyond C^* -algebras the two notions, positive definiteness and complete positivity, still make sense but are no longer equivalent. This happens when one deals with unbounded operator valued functions and moment problems on unbounded sets. Therefore, there is a need of common treatment of these by means of forms over $*$ -semigroups, like in [59]. The aforesaid cases are represented in our paper by unbounded subnormal operators and the complex moment problem. In addition to this we also consider the “bounded” case of unitary dilations of several

1991 *Mathematics Subject Classification.* Primary 43A35, 44A60 Secondary 47A20, 47B20.

Key words and phrases. Positive definite mapping, completely positive mapping, completely f -positive mapping, complex moment problem, multidimensional trigonometric moment problem, truncated moment problems, the 17th Hilbert problem, subnormal operator, unitary power dilation for several contractions.

This work was partially supported by the KBN grant 2 P03A 037 024 and by the MNiSzW grant N201 026 32/1350. The third author also would like to acknowledge an assistance of the EU Sixth Framework Programme for the Transfer of Knowledge “Operator theory methods for differential equations” (TODEQ) # MTKD-CT-2005-030042.

¹ For the reader’s convenience we formulate both the real and the complex variants of the Riesz-Haviland theorem as Theorems A and B in Appendix.

operators and, what is related to, of the multidimensional trigonometric moment problem.

A topic attracting attention of quite a number of mathematicians is extendibility of functions to either positive definite or completely positive ones. The most classical result in this matter concerns groups. It states that every continuous positive definite function on a closed subgroup of a locally compact abelian (or compact, not necessarily abelian) group extends to a continuous positive definite function on the whole group (cf. [32, Section 34.48(a)&(c)] and [16, Theorem 3.16]). However, leaving the topological requirements aside, each positive definite function on a subgroup of a group G extends by zero to a positive definite function on the whole G (cf. [32, Section 32.43(a)]). This procedure is no longer applicable to $*$ -semigroups other than groups. What is more, not every positive definite function on a $*$ -subsemigroup extends to a positive definite function on the whole $*$ -semigroup. This is best exemplified by the interplay between the $*$ -semigroups $\mathfrak{N} \subset \mathfrak{N}_+$ as shown in [55] in connection with the complex moment problem (see Example 3 and Section 5 for the definitions).

The situation becomes more complicated when one wants to extend positive definiteness from subsets of more relaxed structure, even in the case of groups. The classical result of Krein [36] on automatic extendibility of continuous positive definite functions from a symmetric interval to the whole real line suggests that symmetricity of the subset may be essential. This is somehow confirmed by [56] which contains a full characterization of several contractions having commuting unitary dilations. For more discussion of the role played by symmetry we refer to Section 5.2. The results contained in [4, 5] and [15] also corroborate the importance of symmetry, the latter concerns extensions to indefinite forms with finite number of negative squares.

One of the main ideas of the present paper is to employ generalized polynomial functions to the extendibility criteria invented in [55] and [56]. What we get is strictly related to complete positivity of associated linear mappings. The original contribution consists in introducing complete f -positivity (cf. Theorems 14 and 15). This results, in particular, in the complex variant of the Riesz-Haviland theorem (cf. Theorem 20); the complete f -positivity is now written in terms of positivity of the associated linear functional on the set of all finite sums of squares of moduli of very special rational functions in variables z and \bar{z} .

Carefully selected applications are chosen as follows. Considering determining subsets of $\mathfrak{X}_{\mathfrak{N}_+}$ allows us to apply Theorems 14 and 15 to the complex moment problem (Theorem 20) as well as to subnormal operators (Theorem 29). Theorem 20 can be thought of as a truncated moment problem, however not in the usual sense of finite sections. Analogous results are formulated for the truncated multidimensional trigonometric moment problem and the truncated two-sided complex moment problem (cf. Theorems 34 and 40). Moreover, Theorem 32 contains a new characterization of finite systems of Hilbert space operators admitting unitary power dilations.

Section 9 deals, inter alia, with approximation of nonnegative polynomials in indeterminates z and \bar{z} by sums of finitely many squares of moduli of rational functions that are bounded on a neighbourhood of the origin which is assumed to be their only possible singularity (cf. Proposition 36). This can be compared to Artin's solution of the 17th Hilbert problem stating that every nonnegative

polynomial in z and \bar{z} is a sum of finitely many squares of moduli of (a priori arbitrary) rational functions. The above approximation is no longer possible when considered on proper closed subsets of \mathbb{C} , cf. Proposition 24 (see also Proposition 35 for the case of multivariable trigonometric polynomials). Similar approximation holds for multivariable trigonometric polynomials (cf. Proposition 38). A more detailed discussion relating the theme to selected recent articles [22, 29] is contained in Section 9.

GENERAL CRITERIA

Besides keeping $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ for standard sets of integer, real and complex numbers, respectively, by \mathbb{Z}_+ we understand the set $\{0, 1, 2, \dots\}$. Moreover, we adopt the notation $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$ and $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$. As usual, χ_σ stands for the characteristic function of σ , a subset of a set Ω . A system $\{z_\omega\}_{\omega \in \Omega}$ of complex numbers is said to be *finite* if the set $\{\omega \in \Omega: z_\omega \neq 0\}$ is finite.

1. Polynomials on dual $*$ -semigroup. Given a nonempty set Ω , we denote by \mathbb{C}^Ω the complex $*$ -algebra of all complex functions on Ω with the algebra operations defined pointwisely and the involution

$$f^*(\omega) = \overline{f(\omega)}, \quad f \in \mathbb{C}^\Omega, \omega \in \Omega.$$

The following fact reveals the idea standing behind the known characterization of linear independence of Laplace transforms of elements of a commutative semigroup (see [33] and [6, Proposition 6.1.8] for a pattern of the proof).

LEMMA 1. *Let Ω be a nonempty set and let $Y \subset \mathbb{C}^\Omega$ be a semigroup (with respect to the multiplication of \mathbb{C}^Ω) containing the constant function 1. Then the following conditions are equivalent:*

- (i) *Y separates points of Ω (equivalently: if $\widehat{\omega_1}|_Y = \widehat{\omega_2}|_Y$, then $\omega_1 = \omega_2$),*
- (ii) *the system $\{\widehat{\omega}|_Y\}_{\omega \in \Omega}$ is linearly independent in \mathbb{C}^Y ,*

where for $\omega \in \Omega$ the function $\widehat{\omega}: \mathbb{C}^\Omega \rightarrow \mathbb{C}$ is defined by $\widehat{\omega}(f) = f(\omega)$, $f \in \mathbb{C}^\Omega$.

If the Y in Lemma 1 is not semigroup, then the implication (i) \Rightarrow (ii) may not longer be true (cf. Example 3).

As far as abstract semigroups are concerned, we adhere to the multiplicative notation. A mapping $\mathfrak{S} \ni s \mapsto s^* \in \mathfrak{S}$ defined on a semigroup \mathfrak{S} is called an *involution* if $(st)^* = t^*s^*$ and $(s^*)^* = s$ for all $s, t \in \mathfrak{S}$. A semigroup \mathfrak{S} equipped with an involution is said to be a *$*$ -semigroup*. It is clear that if \mathfrak{S} has a unit ε , then $\varepsilon^* = \varepsilon$. The set $\{s \in \mathfrak{S}: s = s^*\}$ of all *Hermitian elements* of a $*$ -semigroup \mathfrak{S} is denoted by \mathfrak{S}_h . For a nonempty subset T of a $*$ -semigroup \mathfrak{S} , we write $T^* = \{s^*: s \in T\}$; T is said to be *symmetric* if $T = T^*$. Put $[T] = \bigcup_{n=1}^\infty T^{[n]}$, where $T^{[n]}$ stands for the set of all products of length n with factors in T . The set $[T]$ is the smallest subsemigroup of \mathfrak{S} containing T . Under the assumption of commutativity of \mathfrak{S} , the set $\llbracket T \rrbracket \stackrel{\text{def}}{=} \{u^*v: u, v \in [T]\}$ is a $*$ -subsemigroup of \mathfrak{S} which does not have to contain any of the sets T and T^* . Neither $[T]$ nor $\llbracket T \rrbracket$ has to contain the unit of \mathfrak{S} even if it exists.

Let \mathfrak{S} be a commutative $*$ -semigroup with a unit ε . A function $\chi: \mathfrak{S} \rightarrow \mathbb{C}$ is called a *character* of \mathfrak{S} if

- (1) $\chi(st) = \chi(s)\chi(t), \quad s, t \in \mathfrak{S},$
- (2) $\chi(s^*) = \overline{\chi(s)}, \quad s \in \mathfrak{S},$

$$(3) \quad \chi(\varepsilon) = 1.$$

The set $\mathfrak{X}_\mathfrak{S}$ of all characters of \mathfrak{S} is a $*$ -semigroup with respect to the multiplication and the involution of $\mathbb{C}^\mathfrak{S}$; $\mathfrak{X}_\mathfrak{S}$ is called a *dual* $*$ -semigroup of \mathfrak{S} . For every $s \in \mathfrak{S}$, we define the function $\hat{s}: \mathfrak{X}_\mathfrak{S} \rightarrow \mathbb{C}$, modelled on the Fourier, Gelfand or Laplace transform, via

$$\hat{s}(\chi) = \chi(s), \quad \chi \in \mathfrak{X}_\mathfrak{S}.$$

The set $\{\hat{s}: s \in \mathfrak{S}\}$ will be denoted by $\widehat{\mathfrak{S}}$. It follows from (1), (2) and (3) that $\widehat{st} = \hat{s} \cdot \hat{t}$ and $\widehat{s^*} = \overline{\hat{s}}$ for all $s, t \in \mathfrak{S}$, and $\widehat{\varepsilon} \equiv 1$. This means that $\widehat{\mathfrak{S}}$ is a $*$ -semigroup with respect to the multiplication and the involution of $\mathbb{C}^{\mathfrak{X}_\mathfrak{S}}$.

The set $\mathcal{F}_\mathfrak{S}$ of all complex functions on \mathfrak{S} vanishing off finite sets is a complex $*$ -algebra with pointwise defined linear algebra operations, the algebra multiplication of convolution type

$$(f \star g)(u) = \sum_{\substack{s, t \in \mathfrak{S} \\ u = st}} f(s)g(t), \quad f, g \in \mathcal{F}_\mathfrak{S}, \quad u \in \mathfrak{S},$$

and involution

$$f^*(s) = \overline{f(s^*)}, \quad f \in \mathcal{F}_\mathfrak{S}, \quad s \in \mathfrak{S}.$$

For a nonempty subset T of \mathfrak{S} , we define the linear subspace $\mathcal{F}_{\mathfrak{S}, T}$ of $\mathcal{F}_\mathfrak{S}$ via

$$\mathcal{F}_{\mathfrak{S}, T} = \{f \in \mathcal{F}_\mathfrak{S} : f \text{ vanishes off the set } T\}.$$

With the algebra operations defined above, $\mathcal{F}_{\mathfrak{S}, T}$ is a subalgebra of $\mathcal{F}_\mathfrak{S}$ if and only if T is a subsemigroup of \mathfrak{S} ; what is more, $\mathcal{F}_{\mathfrak{S}, T}$ is a symmetric subset of $\mathcal{F}_\mathfrak{S}$ if and only if T is a symmetric subset of \mathfrak{S} . The reader should be aware of the fact that if \mathfrak{S} is finite, then though the sets $\mathcal{F}_\mathfrak{S}$ and $\mathbb{C}^\mathfrak{S}$ coincide their $*$ -algebra structures differ unless \mathfrak{S} is a singleton.

Given a nonempty subset Y of $\mathfrak{X}_\mathfrak{S}$, we write $\mathcal{P}(Y)$ for the linear span of $\{\hat{s}|_Y : s \in \mathfrak{S}\}$ in \mathbb{C}^Y . Clearly, $\mathcal{P}(Y)$ is a $*$ -subalgebra of the $*$ -algebra \mathbb{C}^Y . Notice that there exists a unique $*$ -algebra epimorphism $\Delta_Y: \mathcal{F}_\mathfrak{S} \rightarrow \mathcal{P}(Y)$ such that

$$\Delta_Y(\delta_s) = \hat{s}|_Y, \quad s \in \mathfrak{S},$$

where $\delta_s \in \mathcal{F}_\mathfrak{S}$ is the characteristic function of $\{s\}$. For a nonempty subset T of \mathfrak{S} , we write $\mathcal{P}_T(Y) = \Delta_Y(\mathcal{F}_{\mathfrak{S}, T})$; the linear space $\mathcal{P}_T(Y)$ coincides with the linear span of $\{\hat{s}|_Y : s \in T\}$.

In this paper we are interested in the case in which the $*$ -algebra homomorphism Δ_Y is injective. The following is a direct consequence of Lemma 1.

PROPOSITION 2. *If Y is a subsemigroup of $\mathfrak{X}_\mathfrak{S}$ containing the constant function 1, then the following conditions are equivalent*

- (i) Δ_Y is a $*$ -algebra isomorphism,
- (ii) the system $\{\hat{s}|_Y\}_{s \in \mathfrak{S}}$ is linearly independent in \mathbb{C}^Y ,
- (iii) Y separates the points of \mathfrak{S} (equivalently: if $\hat{s}|_Y = \hat{t}|_Y$, then $s = t$).

2. Determining sets. A nonempty subset Y of $\mathfrak{X}_\mathfrak{S}$ is called *determining* (for $\mathcal{P}(\mathfrak{X}_\mathfrak{S})$) if it satisfies any of the equivalent conditions (i) and (ii) of Proposition 2. If a subset Y of $\mathfrak{X}_\mathfrak{S}$ is determining, then the mapping

$$\pi_{T, Y}: \mathcal{P}_T(\mathfrak{X}_\mathfrak{S}) \ni w \mapsto w|_Y \in \mathcal{P}_T(Y)$$

is a well defined $*$ -algebra isomorphism (but not conversely, see the next section).

It may happen that the whole dual $*$ -semigroup $\mathfrak{X}_{\mathfrak{S}}$ does not separate the points of \mathfrak{S} and consequently $\mathfrak{X}_{\mathfrak{S}}$ is not determining (cf. [6, Remarks 4.6.9(1)]). If Y is not a subsemigroup of $\mathfrak{X}_{\mathfrak{S}}$, then implication (iii) \Rightarrow (ii) of Proposition 2 may fail to hold (implication (ii) \Rightarrow (iii) is always true). To see this, we shall discuss a $*$ -semigroup introduced in 1955 by Sz.-Nagy [63] for which the set $\mathcal{P}(\mathfrak{X}_{\mathfrak{S}})$ can be interpreted as the ring of all polynomials in z and \bar{z} .

EXAMPLE 3. Equip the Cartesian product $\mathbb{Z}_+ \times \mathbb{Z}_+$ with coordinatewise defined addition as semigroup operation, i.e. $(i, j) + (k, l) = (i + k, j + l)$, and involution $(m, n)^* = (n, m)$. The $*$ -semigroup so obtained will be denoted by \mathfrak{N} . If \mathbb{C} is thought of as a $*$ -semigroup with multiplication as semigroup operation and complex conjugation as involution, then the mapping

$$\mathfrak{X}_{\mathfrak{N}} \ni \chi \mapsto \chi(1, 0) \in \mathbb{C},$$

being a $*$ -semigroup isomorphism, enables us to identify algebraically $\mathfrak{X}_{\mathfrak{N}}$ with \mathbb{C} . Under this identification, we have

$$(4) \quad \widehat{(m, n)}(z) = z^m \bar{z}^n, \quad m, n \in \mathbb{Z}_+, z \in \mathbb{C}.$$

Note that by (4) and Lemma 5 below the system $\{\widehat{(m, n)}\}_{(m, n) \in \mathfrak{N}}$ is linearly independent in $\mathbb{C}^{\mathbb{C}}$. Hence, the set $\mathcal{P}(\mathfrak{X}_{\mathfrak{N}})$ can be thought of as the ring $\mathbb{C}[z, \bar{z}]$ of all complex polynomials in z and \bar{z} .

It turns out that it is possible to give satisfactory description of all subset of \mathbb{C} separating the points of \mathfrak{N} .

PROPOSITION 4. *A subset Y of \mathbb{C} separates the points of \mathfrak{N} if and only if the following two conditions hold:*

- (i) $Y \not\subset \mathbb{T} \cup \{0\}$,
- (ii) $Y_1 \stackrel{\text{def}}{=} \left\{ \frac{z}{|z|} : z \in Y \setminus \{0\} \right\} \not\subset \{w \in \mathbb{C} : w^{\varkappa} = 1\}$ for every integer $\varkappa \geq 1$.

PROOF. Suppose that Y does not satisfy the conjunction of conditions (i) and (ii). Then either $Y \subset \mathbb{T} \cup \{0\}$, and hence $z^2 \bar{z} = z$ for all $z \in Y$, or $Y_1 \subset \{w \in \mathbb{C} : w^{\varkappa} = 1\}$ for some $\varkappa \geq 1$, and hence $z^{2\varkappa} = z^{\varkappa} \bar{z}^{\varkappa}$ for all $z \in Y$. In both cases, Y cannot separate the points of \mathfrak{N} .

Assume now that Y satisfies (i) and (ii). Take $m, n, k, l \in \mathbb{Z}_+$ and suppose that $z^m \bar{z}^n = z^k \bar{z}^l$ for all $z \in Y$. Taking absolute value of both sides of the equality and employing (i) we get

$$(5) \quad j \stackrel{\text{def}}{=} m + n = k + l.$$

Dividing both sides of $z^m \bar{z}^n = z^k \bar{z}^l$ by $|z|^j$ gives $w^{\varkappa} = 1$ for all $w \in Y_1$ with $\varkappa = m - n - (k - l)$. By (ii) this implies that $\varkappa = 0$, which when combined with (5) leads to $(m, n) = (k, l)$. The proof is complete. \square

We now indicate a class of subsets Y of $\mathfrak{X}_{\mathfrak{N}}$ which separate the points of \mathfrak{N} but which are not determining for $\mathcal{P}(\mathfrak{X}_{\mathfrak{N}}) = \mathbb{C}[z, \bar{z}]$. Take a nonzero polynomial $p \in \mathbb{C}[z, \bar{z}]$ such that the set

$$(6) \quad Y_p \stackrel{\text{def}}{=} \{z \in \mathbb{C} : p(z, \bar{z}) = 0\}$$

is nonempty. Then evidently the functions $\{\widehat{(m, n)}|_{Y_p} : m, n \in \mathbb{Z}_+\}$ are linearly dependent in \mathbb{C}^{Y_p} , and hence Y_p is not determining for $\mathbb{C}[z, \bar{z}]$. Let us focus on the

case of circles and straight lines (which are always of the form (6)). It follows from Proposition 4 that the ensuing sets do not separate the points of \mathfrak{N} :

- the unit circle \mathbb{T} centered at the origin,
- a straight line L such that $0 \in L$ and the points of $L \cap \mathbb{T}$ are complex \varkappa -roots of 1 for some (necessarily even) integer $\varkappa \geq 2$.

All the other circles and straight lines do separate the points of \mathfrak{N} (in many cases they embrace 1). Surprisingly, one point sets, which are still of the form (6), may separate the points of \mathfrak{N} . Indeed, by Proposition 4, if $z \in \mathbb{C} \setminus (\mathbb{T} \cup \{0\})$ and $\frac{z}{|z|}$ is not a complex \varkappa -root of 1 for any integer $\varkappa \geq 1$, then $\{z\}$ does separate the points of \mathfrak{N} .

Contrary to the case of sets separating the points of \mathfrak{N} , one cannot expect any neat description of all determining sets for $\mathbb{C}[z, \bar{z}]$. Nevertheless, we may indicate some determining sets explicitly (see also Lemma 17).

LEMMA 5. *Suppose that $Y \subset \mathbb{C}$ is either a union of infinitely many parallel straight lines or a union of infinitely many concentric circles. If $\{a_{m,n}\}_{m,n \in \mathbb{Z}}$ is a finite system of complex numbers such that $\sum_{m,n \in \mathbb{Z}} a_{m,n} z^m \bar{z}^n = 0$ for all $z \in Y \setminus \{0\}$, then $a_{m,n} = 0$ for all $m, n \in \mathbb{Z}$.*

PROOF. Since the set $\{(m,n) \in \mathbb{Z} \times \mathbb{Z} : a_{m,n} \neq 0\}$ is finite, there exists an integer $N \geq 1$ such that $a_{m,n} = 0$ for all integers m, n such that $m < -N$ or $n < -N$. This implies that $p(z) = \sum_{m,n \geq -N} a_{m,n} z^{m+N} \bar{z}^{n+N}$ is a complex polynomial in z and \bar{z} vanishing on Y . In the case of straight lines, we can always find $\theta \in (-\pi/2, \pi/2]$ such that the polynomial $p(e^{i\theta} z)$ vanishes on a union of infinitely many straight lines parallel to the real axis. Next, considering the complex polynomial $p(e^{i\theta}(x + iy))$ in two real variables x and y , we deduce that $p(z) = 0$ for all $z \in \mathbb{C}$. In the other case, applying a suitable translation of the argument, we can assume that the common center of the circles is in the origin. Employing a well known identity principle for complex polynomials in z and \bar{z} completes the proof in both cases. \square

3. Complete positivity. Let \mathfrak{S} be a unital commutative $*$ -semigroup, T be a nonempty subset of \mathfrak{S} and Y be a nonempty subset of $\mathfrak{X}_{\mathfrak{S}}$. Denote by $\mathcal{P}_T(Y, \ell^2)$ the set of all functions from Y to ℓ^2 of the form $\chi \mapsto \sum_{s \in T} \chi(s) x_s$, where $\{x_s\}_{s \in T} \subset \ell^2$ and $x_s = 0$ for all but a finite number of s 's. Note that $\mathcal{P}_T(Y, \ell^2)$ can be thought of as the algebraic tensor product $\mathcal{P}_T(Y) \otimes \ell^2$. We abbreviate $\mathcal{P}_{\mathfrak{S}}(Y, \ell^2)$ to $\mathcal{P}(Y, \ell^2)$. The standard notation $[a_{k,l}]_{k,l=1}^m \geq 0$ is used for *nonnegativity* of the scalar matrix $[a_{k,l}]_{k,l=1}^m$.

For the linear space

$$M^m(\mathcal{P}_T(Y)) \stackrel{\text{def}}{=} \{[w_{k,l}]_{k,l=1}^m : w_{k,l} \in \mathcal{P}_T(Y) \text{ for all } k, l = 1, \dots, m\}, \quad m \geq 1,$$

its subsets

$$M_+^m(\mathcal{P}_T(Y)) \stackrel{\text{def}}{=} \{[w_{k,l}]_{k,l=1}^m \in M^m(\mathcal{P}_T(Y)) : [w_{k,l}(\chi)]_{k,l=1}^m \geq 0 \text{ for all } \chi \in Y\}$$

and

$$M_f^m(\mathcal{P}_T(Y)) \stackrel{\text{def}}{=} M^m(\mathcal{P}_T(Y)) \cap \{[\langle p_k, p_l \rangle]_{k,l=1}^m : p_1, \dots, p_m \in \mathcal{P}(Y, \ell^2)\},$$

where $\langle p_k, p_l \rangle$ is the function $\chi \mapsto \langle p_k(\chi), p_l(\chi) \rangle_{\ell^2}$, turn out to be convex cones. Note that if $p_1, \dots, p_m \in \mathcal{P}(Y, \ell^2)$, then $[\langle p_k, p_l \rangle]_{k,l=1}^m \in M_+^m(\mathcal{P}(Y))$. This implies that $M_f^m(\mathcal{P}_T(Y)) \subset M_+^m(\mathcal{P}_T(Y))$. One can also check that the square of each

matrix from $M_f^m(\mathcal{P}_T(Y))$ (respectively $M_+^m(\mathcal{P}_T(Y))$) belongs to $M_f^m(\mathcal{P}_{T[2]}(Y))$ (respectively $M_+^m(\mathcal{P}_{T[2]}(Y))$). What is more, if Y is a determining subset of $\mathfrak{X}_\mathfrak{S}$, then the mappings

$$(7) \quad \pi_{T,Y}^{(m)}: M^m(\mathcal{P}_T(\mathfrak{X}_\mathfrak{S})) \ni [w_{k,l}]_{k,l=1}^m \mapsto [w_{k,l}|_Y]_{k,l=1}^m \in M^m(\mathcal{P}_T(Y)),$$

are linear isomorphisms and consequently

$$(8) \quad \pi_{T,Y}^{(m)}(M_f^m(\mathcal{P}_T(\mathfrak{X}_\mathfrak{S}))) = M_f^m(\mathcal{P}_T(Y)).$$

LEMMA 6. *If $p_1, \dots, p_m \in \mathcal{P}(Y, \ell^2)$ and $A = [\langle p_k, p_l \rangle]_{k,l=1}^m$, then there exists a matrix $P = [p_{k,j}]_{k=1}^m_{j=1}^n$ with entries $p_{k,j} \in \mathcal{P}(Y)$ such that $A = PP^*$, where $P^* = [\bar{p}_{j,k}]_{k=1}^m_{j=1}^n$. Conversely, if P is a matrix of size $m \times n$ with entries in $\mathcal{P}(Y)$, then $PP^* = [\langle p_k, p_l \rangle]_{k,l=1}^m$ with some $p_1, \dots, p_m \in \mathcal{P}(Y, \ell^2)$. In particular, if $p \in \mathcal{P}(Y, \ell^2)$, then $\langle p, p \rangle = \sum_{j=1}^n |q_j|^2$ with some $q_j \in \mathcal{P}(Y)$.*

PROOF. Take $p_1, \dots, p_m \in \mathcal{P}(Y, \ell^2)$. Then there exists a finite orthonormal basis $\{e_j\}_{j=1}^n$ of the linear span of $\bigcup_{k=1}^m p_k(Y)$. As a consequence, $p_k = \sum_{j=1}^n p_{k,j} e_j$ with some $p_{k,j} \in \mathcal{P}(Y)$, and hence $P = [p_{k,j}]_{k=1}^m_{j=1}^n$ is the required matrix. Reversing the above reasoning concludes the proof. \square

We now examine the behavior of the classes $M_f^m(\mathcal{P}_T(Y))$ and $M_+^m(\mathcal{P}_T(Y))$ under the operation of transposing their members.

LEMMA 7. (i) *If a matrix $[w_{k,l}]_{k,l=1}^m$ is a member of $M_+^m(\mathcal{P}_T(Y))$, then the transposed matrix $[w_{l,k}]_{k,l=1}^m$ belongs to $M_+^m(\mathcal{P}_{T^*}(Y))$.*

(ii) *If $p_1, \dots, p_m \in \mathcal{P}_T(Y, \ell^2)$, then there exist $q_1, \dots, q_m \in \mathcal{P}_{T^*}(Y, \ell^2)$ such that*

$$(9) \quad \langle p_l, p_k \rangle = \langle q_k, q_l \rangle, \quad k, l = 1, \dots, m.$$

PROOF. (i) By nonnegativity of $[w_{k,l}]_{k,l=1}^m$, we have $w_{l,k}(\chi) = \overline{w_{k,l}(\chi)}$ for all $\chi \in Y$ and $k, l = 1, \dots, m$. Hence, the application of (2) justifies (i).

(ii) Arguing as in the proof of Lemma 6, we can write $p_k = \sum_{j=1}^n p_{k,j} e_j$ with some $p_{k,j} \in \mathcal{P}_T(Y)$. Then

$$\langle p_l, p_k \rangle(\chi) = \sum_{j=1}^n p_{l,j}(\chi) \overline{p_{k,j}(\chi)} = \langle q_k, q_l \rangle(\chi), \quad \chi \in Y,$$

where $q_k(\chi) = \sum_{j=1}^n \overline{p_{k,j}(\chi)} e_j$ for $\chi \in Y$. By (2), the functions $Y \ni \chi \mapsto \overline{p_{k,j}(\chi)} \in \mathbb{C}$ belong to $\mathcal{P}_{T^*}(Y)$. This completes the proof. \square

Let F be a topological linear space and τ be its topology. The closure of a subset W of F with respect to τ is denoted by \overline{W}^τ . Given an integer $m \geq 1$, we write $M^m(F)$ for the linear space (with entrywise defined linear operations) of all m by m matrices with entries in F . Identifying $M^m(F)$ with the Cartesian product of m^2 copies of F , we may regard $M^m(F)$ as a topological linear space with the product topology $\tau^{(m)}$.

Call a nonempty subset Z of $\mathfrak{X}_\mathfrak{S}$ T -bounded² if $\sup_{\chi \in Z} |\chi(s)| < \infty$ for every $s \in T$. It is obvious that a nonempty subset Z of $\mathfrak{X}_\mathfrak{S}$ is T -bounded if and only if it is $\langle T \rangle$ -bounded, where $\langle T \rangle$ stands for the unital $*$ -semigroup generated by T . Note also

² Note that in view of Tychonoff's theorem \mathfrak{S} -bounded sets coincide with subsets of $\mathfrak{X}_\mathfrak{S}$ which are relatively compact in the topology of pointwise convergence on \mathfrak{S} .

that nonempty $Z \subset \mathfrak{X}_{\mathfrak{S}}$ is T -bounded if and only if for every (equivalently: for some) integer $m \geq 1$ and for every $w \in M^m(\mathcal{P}_T(Z))$, $\sup_{\chi \in Z} \|w(\chi)\| < \infty$; here $\|w(\chi)\|$ stands for the operator norm of the matrix $w(\chi)$. Denote by $\tau_{T,Y}$ the locally convex topology on $\mathcal{P}_T(Y)$ given by the system of seminorms $w \mapsto \sup_{\chi \in Z} |w(\chi)|$ indexed by T -bounded subsets Z of Y . Observe that the topology $\tau_{T,Y}^{(m)}$ on $M^m(\mathcal{P}_T(Y))$ is identical with the locally convex topology given by the system of seminorms $w \mapsto \sup_{\chi \in Z} \|w(\chi)\|$ with Z ranging over all T -bounded subsets of Y . It is clear that the topology $\tau_{T,Y}^{(m)}$ is stronger than the topology of pointwise convergence on Y . If $\mathfrak{X}_{\mathfrak{S}}$ is equipped with the topology of pointwise convergence on \mathfrak{S} , then $\tau_{T,Y}^{(m)}$ is still stronger than the topology of uniform convergence on compact subsets of Y (see Footnote 2). In turn, if there exist integers $i, j, k, l \geq 0$ such that $i + j \neq k + l$ and

$$(10) \quad s^{*i} s^j = s^{*k} s^l, \quad s \in T,$$

then $\sup_{\chi \in \mathfrak{X}_{\mathfrak{S}}} |\chi(s)| \leq 1$ for every $s \in T$. Hence, in this specific situation, the topology $\tau_{T,Y}^{(m)}$ describes exactly the uniform convergence on Y . Property (10) holds for any T which is a unital commutative inverse semigroup (take $i = l = 1$, $j = 2$ and $k = 0$, cf. [20]); in particular, this is the case for T being an abelian group with involution $s^* = s^{-1}$.

Below we stick to the notations declared at the beginning of this section.

LEMMA 8. $M_f^m(\mathcal{P}_T(Y)) \subset M_+^m(\mathcal{P}_T(Y)) \subset \overline{M_f^m(\mathcal{P}_{[T]}(Y))}^\tau$; $\tau = \tau_{[T],Y}^{(m)}$.

PROOF. Only the second inclusion has to be justified. Consider a $[T]$ -bounded subset Z of Y and take $w = [w_{k,l}]_{k,l=1}^m \in M_+^m(\mathcal{P}_T(Y))$. By Lemma 7, the transpose v of w is a member of $M_+^m(\mathcal{P}_{T^*}(Y))$. Note also that $\|v(\chi)\| = \|w(\chi)\|$ for every $\chi \in Y$, and consequently, $M \stackrel{\text{def}}{=} \sup_{\chi \in Z} \|v(\chi)\| < \infty$. Without loss of generality we can assume that $M > 0$. By the Weierstrass theorem, there exists a sequence of real polynomials $\{\rho_n\}_{n=1}^\infty$ vanishing at 0 which tends to the square root function uniformly on $[0, M]$. For $\chi \in Y$, we denote by $\sqrt{v(\chi)}$ the square root of $v(\chi)$. As for every $\chi \in Z$ the norm of the nonnegative matrix $v(\chi)$ is less than or equal to M , we get

$$(11) \quad \sup_{\chi \in Z} \|\sqrt{v(\chi)} - \rho_n(v(\chi))\| \leq \sup_{x \in [0, M]} |\sqrt{x} - \rho_n(x)|, \quad n \geq 1.$$

Let $\{e_k\}_{k=1}^m$ be the standard ‘0-1’ basis of \mathbb{C}^m . Write $\rho_n(v)e_k$ for the function $Y \ni \chi \rightarrow \rho_n(v(\chi))e_k \in \ell^2$ (after a natural identification). Since $\rho_n(0) = 0$, we see that $\rho_n(v)e_k \in \mathcal{P}_{[T^*]}(Y, \ell^2)$ and hence $[\langle \rho_n(v)e_k, \rho_n(v)e_l \rangle]_{k,l=1}^m \in M_f^m(\mathcal{P}_{[T]}(Y))$. It follows from (11) that $[\langle \rho_n(v)e_k, \rho_n(v)e_l \rangle]_{k,l=1}^m$ converges uniformly on Z to $\tilde{v} = [\tilde{v}_{k,l}]_{k,l=1}^m$ as $n \rightarrow \infty$, where $\tilde{v}_{k,l}(\chi) \stackrel{\text{def}}{=} \langle \sqrt{v(\chi)}e_k, \sqrt{v(\chi)}e_l \rangle$ for $\chi \in Y$. Since

$$\tilde{v}_{k,l}(\chi) = \langle v(\chi)e_k, e_l \rangle = v_{l,k}(\chi) = w_{k,l}(\chi), \quad \chi \in Y, \quad k, l = 1, \dots, m,$$

and the class of $[T]$ -bounded subsets of Y is directed upwards by inclusion, the proof is complete. \square

REMARK 9. By the proof of Lemma 8, every matrix in $M_+^m(\mathcal{P}_T(Y))$ can be approximated in the topology $\tau_{[T],Y}^{(m)}$ by means of matrices of the form $[\langle p_k, p_l \rangle]_{k,l=1}^m \in M_f^m(\mathcal{P}_{[T]}(Y))$, where $p_1, \dots, p_m \in \mathcal{P}_{[T^*]}(Y, \ell^2)$.

Let \mathcal{D} be a complex linear space. Denote by $\mathcal{S}(\mathcal{D})$ the set of all sesquilinear forms on \mathcal{D} . For every integer $m \geq 1$, we define:

$$\begin{aligned} M^m(\mathcal{S}(\mathcal{D})) &= \{[\phi_{k,l}]_{k,l=1}^m : \phi_{k,l} \in \mathcal{S}(\mathcal{D}) \text{ for all } k, l = 1, \dots, m\}, \\ M_+^m(\mathcal{S}(\mathcal{D})) &= \{[\phi_{k,l}]_{k,l=1}^m \in M^m(\mathcal{S}(\mathcal{D})) : [\phi_{k,l}]_{k,l=1}^m \gg 0\}, \end{aligned}$$

where the notation $[\phi_{k,l}]_{k,l=1}^m \gg 0$ means that

$$\sum_{k,l=1}^m \phi_{k,l}(h_k, h_l) \geq 0 \quad \text{for all } h_1, \dots, h_m \in \mathcal{D}.$$

We say that a mapping $\Psi: \mathfrak{S} \rightarrow \mathcal{S}(\mathcal{D})$ is *positive definite* if

$$\sum_{k,l=1}^m \Psi(s_l^* s_k)(h_k, h_l) \geq 0$$

for every integer $m \geq 1$ and for all $s_1, \dots, s_m \in \mathfrak{S}$ and $h_1, \dots, h_m \in \mathcal{D}$.

Suppose that Y is a determining subset of $\mathfrak{X}_{\mathfrak{S}}$. Then for every mapping $\Phi: T \rightarrow \mathcal{S}(\mathcal{D})$, there exists a unique linear mapping $\Lambda_{\Phi,Y}: \mathcal{P}_T(Y) \rightarrow \mathcal{S}(\mathcal{D})$ such that

$$(12) \quad \Lambda_{\Phi,Y}(\hat{s}|_Y) = \Phi(s), \quad s \in T.$$

We say that the mapping $\Lambda_{\Phi,Y}$ is *completely positive* if for every integer $m \geq 1$,

$$\Lambda_{\Phi,Y}^{(m)}(M_+^m(\mathcal{P}_T(Y))) \subset M_+^m(\mathcal{S}(\mathcal{D})),$$

where $\Lambda_{\Phi,Y}^{(m)}: M^m(\mathcal{P}_T(Y)) \rightarrow M^m(\mathcal{S}(\mathcal{D}))$ is a linear mapping defined by

$$\Lambda_{\Phi,Y}^{(m)}([w_{k,l}]_{k,l=1}^m) = [\Lambda_{\Phi,Y}(w_{k,l})]_{k,l=1}^m, \quad [w_{k,l}]_{k,l=1}^m \in M^m(\mathcal{P}_T(Y)).$$

$\Lambda_{\Phi,Y}$ is said to be *completely f-positive* if for every integer $m \geq 1$,

$$\Lambda_{\Phi,Y}^{(m)}(M_f^m(\mathcal{P}_T(Y))) \subset M_+^m(\mathcal{S}(\mathcal{D})).$$

Apparently, complete positivity implies complete f-positivity. If $Y = \mathfrak{X}_{\mathfrak{S}}$, we shall abbreviate $\Lambda_{\Phi,Y}$ ($\Lambda_{\Phi,Y}^{(m)}$ respectively) to Λ_{Φ} ($\Lambda_{\Phi}^{(m)}$ respectively).

We now show that the notion of complete f-positivity of $\Lambda_{\Phi,Y}$ does not depend on the choice of determining set Y .

PROPOSITION 10. *Suppose that \mathfrak{S} is a unital commutative $*$ -semigroup and T is a nonempty subset of \mathfrak{S} . Let \mathcal{D} be a complex linear space and $\Phi: T \rightarrow \mathcal{S}(\mathcal{D})$ be a mapping. If Y and Z are determining subsets of $\mathfrak{X}_{\mathfrak{S}}$, then $\Lambda_{\Phi,Y}$ is completely f-positive if and only if $\Lambda_{\Phi,Z}$ is completely f-positive.*

PROOF. The proof reduces to the case $Z = \mathfrak{X}_{\mathfrak{S}}$. By (7) and (12) we have

$$\Lambda_{\Phi,Y}^{(m)} \circ \pi_{T,Y}^{(m)} = \Lambda_{\Phi}^{(m)}, \quad m = 1, 2, \dots,$$

which, together with (8), implies the desired equivalence. \square

For a complex linear space \mathcal{D} , we denote by $\varrho_{\mathcal{D}}$ the locally convex topology on $\mathcal{S}(\mathcal{D})$ given by the system of seminorms $\phi \mapsto |\phi(f, g)|$ indexed by all the pairs $(f, g) \in \mathcal{D} \times \mathcal{D}$. Clearly, the topology $\varrho_{\mathcal{D}}$ is nothing else than that of pointwise convergence on $\mathcal{D} \times \mathcal{D}$, and therefore it can be regarded as an analogue of the weak operator topology on the set of bounded linear operators on a Hilbert space.

PROPOSITION 11. *Assume that \mathfrak{S} is a unital commutative $*$ -semigroup, T is a nonempty subset of \mathfrak{S} such that $\llbracket T \rrbracket \subset T$, and Y is a determining subsets of $\mathfrak{X}_{\mathfrak{S}}$. Let \mathcal{D} be a complex linear space and $\Phi: T \rightarrow \mathcal{S}(\mathcal{D})$ be a mapping. Suppose that the mapping $\Lambda_{\Phi,Y}: (\mathcal{P}_T(Y), \tau_{T,Y}) \rightarrow (\mathcal{S}(\mathcal{D}), \varrho_{\mathcal{D}})$ is continuous. Then $\Lambda_{\Phi,Y}$ is completely f -positive if and only if $\Lambda_{\Phi,Y}$ is completely positive.*

PROOF. One can check that the assumed continuity of $\Lambda_{\Phi,Y}$ implies that of

$$\Lambda_{\Phi,Y}^{(m)}: (M^m(\mathcal{P}_T(Y)), \tau_{T,Y}^{(m)}) \rightarrow (M^m(\mathcal{S}(\mathcal{D})), \varrho_{\mathcal{D}}^{(m)}), \quad m = 1, 2, \dots$$

Hence, by the $\varrho_{\mathcal{D}}^{(m)}$ -closedness of $M_+^m(\mathcal{S}(\mathcal{D}))$ and Lemma 8, we arrive at the desired conclusion. \square

Regarding Proposition 11, note that if T is a $*$ -subsemigroup of a unital commutative $*$ -semigroup \mathfrak{S} (T need not be unital), then $\llbracket T \rrbracket \subset T$. The reverse implication is not true in general. In fact, it may happen that $\llbracket T \rrbracket \subset T$ although T is not a subsemigroup of \mathfrak{S} and $T \neq T^*$ (thus neither $T \subset \llbracket T \rrbracket$ nor $T^* \subset \llbracket T \rrbracket$ holds). We leave it to the reader to verify that this is the case for the subset

$$T \stackrel{\text{def}}{=} \{(k, l) \in \mathfrak{N}: k \geq 1, l \geq 1\} \cup T_0$$

of the $*$ -semigroup \mathfrak{N} considered in Example 3, where T_0 is a proper subset of $\{(k, 0) \in \mathfrak{N}: k \geq 1\}$ containing $(1, 0)$.

4. Criteria for extendibility. As mentioned in Introduction, not every positive definite function on \mathfrak{N} extends to a positive definite function on the $*$ -semigroup \mathfrak{N}_+ (see Example 3 and Section 5 for the definitions). However, if we impose a stronger positivity-like condition on a function defined on \mathfrak{N} (in the language of [11] this is positive definiteness with respect to \mathfrak{N}_+), then it is extendable to a positive definite function on \mathfrak{N}_+ , and reversely (cf. [55]). The property of extendibility of positive definite functions was characterized likewise in [55] also in the case of $*$ -subsemigroups of abstract (unital commutative) $*$ -semigroups. This led the authors of [55] to find a new solution of the complex moment problem (not to mention other extension results). The key feature that made this approach successful was the semiperfectness of \mathfrak{N}_+ , the property guaranteeing that every positive definite function on \mathfrak{N}_+ is a moment function, i.e. it has the Laplace-type integral representation on the dual $*$ -semigroup of \mathfrak{N}_+ . Inspired by this, Bisgaard attached to any $*$ -semigroup \mathfrak{S} an enveloping perfect³ $*$ -semigroup \mathfrak{Q} such that the set of all moment functions on \mathfrak{S} coincides with the set of all functions which are positive definite with respect to \mathfrak{Q} (cf. [11]). The instance of semiperfect (but not perfect) $*$ -semigroup \mathfrak{N}_+ as an extending $*$ -semigroup for \mathfrak{N} shows that semiperfectness is sufficient as far as moment problems are concerned. Though there is a limited freedom of choice of an extending semiperfect $*$ -semigroup for a fixed $*$ -semigroup, it can by no means be chosen arbitrarily, as indicated in the discussion concerning the inclusions (22) in [55]. In this section we are looking for criteria that guarantee positive definite extendibility of mappings defined on symmetric subsets of (operator) semiperfect $*$ -semigroups. As a result, we obtain the characterizations of “truncated” moment functions enriching those in [55, 56] with the new positivity conditions of the Riesz-Haviland type.

³ i.e. semiperfect with the uniqueness of integral representation

Suppose that \mathbf{M} is a σ -algebra of subsets of a set $X \neq \emptyset$ and \mathcal{D} is a complex linear space. A mapping $\mu: \mathbf{M} \rightarrow \mathcal{S}(\mathcal{D})$ is called a *semispectral measure*⁴ on \mathbf{M} if $\mu(\cdot)(f, f)$ is a finite positive measure for every $f \in \mathcal{D}$.

Let \mathfrak{S} be a unital commutative $*$ -semigroup. Denote by $\mathbf{M}_{\mathfrak{S}}$ the smallest σ -algebra of subsets of $\mathfrak{X}_{\mathfrak{S}}$ with respect to which all the transforms \hat{s} , $s \in \mathfrak{S}$, are measurable. Following [8], we say that \mathfrak{S} is *operator semiperfect* if for any complex linear space \mathcal{D} and for any positive definite mapping $\Psi: \mathfrak{S} \rightarrow \mathcal{S}(\mathcal{D})$ there exists a semispectral measure $\mu: \mathbf{M}_{\mathfrak{S}} \rightarrow \mathcal{S}(\mathcal{D})$ such that

$$(13) \quad \Psi(s) = \int_{\mathfrak{X}_{\mathfrak{S}}} \hat{s}(\chi) \mu(d\chi), \quad s \in \mathfrak{S}.$$

This equality is to be understood in the following sense

$$\Psi(s)(f, g) = \int_{\mathfrak{X}_{\mathfrak{S}}} \hat{s}(\chi) \mu(d\chi)(f, g), \quad f, g \in \mathcal{D}, \quad s \in \mathfrak{S};$$

here and forth all the integrands are tacitly assumed to be summable. An equivalent definition of operator semiperfectness may be stated in a matrix-type form, as shown in [8].

Note that if $\mathfrak{X}_{\mathfrak{S}}$ is equipped with the topology of pointwise convergence on \mathfrak{S} , then the transforms \hat{s} , $s \in \mathfrak{S}$, are continuous, and consequently the σ -algebra $\mathbf{M}_{\mathfrak{S}}$ is contained in the σ -algebra of all Borel subsets of $\mathfrak{X}_{\mathfrak{S}}$ (the equality does not hold in general). It is also clear that if \mathbf{M} is a σ -algebra of subsets of $\mathfrak{X}_{\mathfrak{S}}$ such that $\mathbf{M}_{\mathfrak{S}} \subset \mathbf{M}$ and μ is a semispectral measure on \mathbf{M} satisfying (13), then the restriction of μ to $\mathbf{M}_{\mathfrak{S}}$ satisfies (13) as well.

LEMMA 12. *Let \mathfrak{S} be a unital commutative $*$ -semigroup whose dual $*$ -semigroup $\mathfrak{X}_{\mathfrak{S}}$ is determining and let T be a nonempty subset of \mathfrak{S} . Suppose that a complex linear space \mathcal{D} and a mapping $\Phi: T \rightarrow \mathcal{S}(\mathcal{D})$ are given.*

- (i) *If $\mu: \mathbf{M} \rightarrow \mathcal{S}(\mathcal{D})$ is a semispectral measure on a σ -algebra \mathbf{M} of subsets of $\mathfrak{X}_{\mathfrak{S}}$, $\mathbf{M}_{\mathfrak{S}} \subset \mathbf{M}$ and Y is a determining subset of $\mathfrak{X}_{\mathfrak{S}}$ such that $Y \in \mathbf{M}$ and*

$$\Phi(s) = \int_Y \hat{s} d\mu, \quad s \in T,$$

then $\Lambda_{\Phi, Y}$ is completely positive.

- (ii) *If $T = \mathfrak{S}$ and Λ_{Φ} is completely f -positive, then Φ is positive definite.*

In particular, if \mathfrak{S} is operator semiperfect and $T = \mathfrak{S}$, then Φ is positive definite if and only if Λ_{Φ} is completely f -positive or equivalently if Λ_{Φ} is completely positive.

PROOF. (i) Take $[w_{k,l}]_{k,l=1}^m \in \mathbf{M}_+^m(\mathcal{P}_T(Y))$ and $e_1, \dots, e_m \in \mathcal{D}$. Notice that the complex measures $\mu(\cdot)(e_i, e_j)$, $i, j \in \{1, \dots, m\}$, are absolutely continuous with respect to some finite positive measure ν on \mathbf{M} (e.g. $\nu(\cdot) = \sum_{i=1}^m \mu(\cdot)(e_i, e_i)$). By the Radon-Nikodym theorem, there exists a system $\{h_{i,j}\}_{i,j=1}^m$ of \mathbf{M} -measurable complex functions on $\mathfrak{X}_{\mathfrak{S}}$ such that $\mu(\sigma)(e_i, e_j) = \int_{\sigma} h_{i,j} d\nu$ for all $\sigma \in \mathbf{M}$. Since $[\mu(\sigma)(e_i, e_j)]_{i,j=1}^m \geq 0$ for every $\sigma \in \mathbf{M}$, one can show (see the proof of [39, Theorem 6.4]) that there exists $Z \in \mathbf{M}$ such that $\nu(\mathfrak{X}_{\mathfrak{S}} \setminus Z) = 0$ and $[h_{k,l}(\chi)]_{k,l=1}^m \geq 0$ for

⁴ With the natural identification of bounded linear operators with sesquilinear forms, our definition subsumes the classical semispectral operator-valued measures (cf. [39]).

every $\chi \in Z$. Let for $\chi \in Z$, $[a_{k,l}(\chi)]_{k,l=1}^m$ be the square root of $[h_{k,l}(\chi)]_{k,l=1}^m$. Since $[w_{k,l}(\chi)]_{k,l=1}^m \geq 0$ for every $\chi \in Y$ and $\nu(Y \setminus (Y \cap Z)) = 0$, we get

$$\sum_{k,l=1}^m \Lambda_{\Phi,Y}(w_{k,l})(e_k, e_l) = \int_{Y \cap Z} \sum_{k,l=1}^m w_{k,l} h_{k,l} d\nu = \int_{Y \cap Z} \sum_{j=1}^m \sum_{k,l=1}^m w_{k,l} a_{k,j} \overline{a_{l,j}} d\nu \geq 0.$$

(ii) Take finite sequences $s_1, \dots, s_m \in \mathfrak{S}$ and $e_1, \dots, e_m \in \mathcal{D}$. It is easily seen that $[\widehat{s_k s_l^*}]_{k,l=1}^m \in M_f^m(\mathcal{P}(\mathfrak{X}_{\mathfrak{S}}))$. By complete f -positivity of Λ_{Φ} , we get $\sum_{k,l=1}^m \Phi(s_l^* s_k)(e_k, e_l) \geq 0$, which finishes the proof. \square

We now deal with the question of when a $\mathcal{S}(\mathcal{D})$ -valued mapping extends from a subset T of \mathfrak{S} to a positive definite mapping on the whole of \mathfrak{S} . The following result is the main tool in our considerations. It is proved as a consequence of Theorem 1 of [56] (in fact, a prototype of this theorem appeared in [55]). Below, we interpret the algebraic tensor product $\mathcal{F}_{\mathfrak{S}} \otimes \ell^2$ as the collection of all ℓ^2 -valued functions on \mathfrak{S} vanishing off finite sets. Revoking the definition of $\mathfrak{S}_{\mathfrak{h}}$ from page 3 it may be convenient for further references to detach the following condition

(14) T is a symmetric subset of \mathfrak{S} containing $\mathfrak{S}_{\mathfrak{h}}$.

LEMMA 13. *Suppose that \mathfrak{S} is a unital commutative $*$ -semigroup, T satisfies (14) and Y is a determining subset of $\mathfrak{X}_{\mathfrak{S}}$. If \mathcal{D} is a complex linear space, then for every mapping $\Phi: T \rightarrow \mathcal{S}(\mathcal{D})$ the following three conditions are equivalent:*

- (i) Φ extends to a positive definite mapping $\Psi: \mathfrak{S} \rightarrow \mathcal{S}(\mathcal{D})$,
 - (ii) $\sum_{k,l=1}^m \sum_{\substack{s,t \in \mathfrak{S} \\ t^* s \in T}} \Phi(t^* s)(e_k, e_l) \langle \lambda_k(s), \lambda_l(t) \rangle_{\ell^2} \geq 0$ for every integer $m \geq 1$ and for all finite systems $\{e_n\}_{n=1}^m \subset \mathcal{D}$ and $\{\lambda_n\}_{n=1}^m \subset \mathcal{F}_{\mathfrak{S}} \otimes \ell^2$ such that
- $$(15) \quad \sum_{\substack{s,t \in \mathfrak{S} \\ t^* s = u}} \langle \lambda_i(s), \lambda_j(t) \rangle_{\ell^2} = 0, \quad u \in \mathfrak{S} \setminus T, i, j = 1, \dots, m,$$
- (iii) $\Lambda_{\Phi,Y}$ is completely f -positive.

PROOF. (i) \Leftrightarrow (ii) The reader can convince himself that condition (ii) of [56, Theorem 1] is equivalent to our condition (ii); note that this can be shown directly, without recourse to the proof given therein. Hence, our equivalence (i) \Leftrightarrow (ii) is a consequence of [56, Theorem 1].

(ii) \Rightarrow (iii) Fix an integer $m \geq 1$. Take $w = [\langle p_k, p_l \rangle]_{k,l=1}^m \in M_f^m(\mathcal{P}_T(Y))$ with $p_1, \dots, p_m \in \mathcal{P}(Y, \ell^2)$. Then there exist $\lambda_1, \dots, \lambda_m \in \mathcal{F}_{\mathfrak{S}} \otimes \ell^2$ such that

$$(16) \quad p_k(\chi) = \sum_{s \in \mathfrak{S}} \chi(s) \lambda_k(s), \quad \chi \in Y, k = 1, \dots, m.$$

It is clear that

$$(17) \quad \langle p_k(\chi), p_l(\chi) \rangle_{\ell^2} = \sum_{u \in \mathfrak{S}} \chi(u) \sum_{\substack{s,t \in \mathfrak{S} \\ t^* s = u}} \langle \lambda_k(s), \lambda_l(t) \rangle_{\ell^2}, \quad \chi \in Y, k, l = 1, \dots, m.$$

Since $w \in M^m(\mathcal{P}_T(Y))$ and Y is determining, we see that (17) implies (15). It follows from (12), (15), (17) and (ii) that

$$\sum_{k,l=1}^m \Lambda_{\Phi,Y}(\langle p_k, p_l \rangle)(e_k, e_l) = \sum_{k,l=1}^m \sum_{u \in T} \sum_{\substack{s,t \in \mathfrak{S} \\ t^* s = u}} \Phi(u)(e_k, e_l) \langle \lambda_k(s), \lambda_l(t) \rangle_{\ell^2}$$

$$= \sum_{k,l=1}^m \sum_{\substack{s,t \in \mathfrak{S} \\ t^* s \in T}} \Phi(t^* s)(e_k, e_l) \langle \lambda_k(s), \lambda_l(t) \rangle_{\ell^2} \geq 0$$

for all vectors $e_1, \dots, e_m \in \mathcal{D}$. This shows that $\Lambda_{\Phi, Y}$ is completely f-positive.

Reversing the above reasoning, we infer (ii) from (iii); to see this, for fixed $\lambda_1, \dots, \lambda_m \in \mathcal{F}_{\mathfrak{S}} \otimes \ell^2$ consider $p_1, \dots, p_m \in \mathcal{P}(Y, \ell^2)$ defined by (16). This completes the proof. \square

With the above discussions, we are in a position to state the main result of the paper which supplies criteria for extendibility to a positive definite function.

THEOREM 14. *Suppose that \mathfrak{S} is an operator semiperfect $*$ -semigroup, T satisfies (14) and Y is a determining subset of $\mathfrak{X}_{\mathfrak{S}}$. If \mathcal{D} is a complex linear space, then for every mapping $\Phi: T \rightarrow \mathcal{S}(\mathcal{D})$ the following four conditions are equivalent:*

- (i) Φ extends to a positive definite mapping $\Psi: \mathfrak{S} \rightarrow \mathcal{S}(\mathcal{D})$,
- (ii) $\Phi(s) = \int_{\mathfrak{X}_{\mathfrak{S}}} \hat{s} d\mu$ for all $s \in T$ with some semispectral measure μ on $\mathbf{M}_{\mathfrak{S}}$,
- (iii) Λ_{Φ} is completely positive⁵,
- (iv) $\Lambda_{\Phi, Y}$ is completely f-positive.

PROOF. (i) \Rightarrow (ii) Use operator semiperfectness of \mathfrak{S} .

(ii) \Rightarrow (i) Since $|\hat{s}|^2 = \widehat{s^* s}$ and $s^* s \in T$ for every $s \in \mathfrak{S}$, we see that the function \hat{s} is square summable with respect to μ for every $s \in \mathfrak{S}$. This enables us to define the mapping $\Psi: \mathfrak{S} \rightarrow \mathcal{S}(\mathcal{D})$ by

$$\Psi(s) = \int_{\mathfrak{X}_{\mathfrak{S}}} \hat{s} d\mu, \quad s \in \mathfrak{S},$$

It follows from Lemma 12 that Ψ is a positive definite extension of Φ .

(i) \Leftrightarrow (iv) Apply Lemma 13.

(ii) \Rightarrow (iii) This is a consequence of Lemma 12.

(iii) \Rightarrow (iv) Since (iii) implies (iv) with $Y = \mathfrak{X}_{\mathfrak{S}}$, we see that Λ_{Φ} is completely f-positive. An application of Proposition 10 guarantees that $\Lambda_{\Phi, Y}$ is completely f-positive as well. This completes the proof. \square

We now turn to the the case of scalar functions. A unital commutative $*$ -semigroup \mathfrak{S} is called *semiperfect* (cf. [8]) if for any positive definite function $\Psi: \mathfrak{S} \rightarrow \mathbb{C}$ there exists a finite positive measure μ on $\mathbf{M}_{\mathfrak{S}}$ such that

$$\Psi(s) = \int_{\mathfrak{X}_{\mathfrak{S}}} \hat{s}(\chi) \mu(d\chi), \quad s \in \mathfrak{S}.$$

Evidently, operator semiperfectness implies semiperfectness but not conversely as indicated by Bisgaard in [12]. The following is a scalar counterpart of Theorem 14.

THEOREM 15. *Suppose that \mathfrak{S} is a semiperfect $*$ -semigroup, T satisfies (14) and Y is a determining subset of $\mathfrak{X}_{\mathfrak{S}}$. Then for every function $\varphi: T \rightarrow \mathbb{C}$ the following four conditions are equivalent:*

- (i) φ extends to a positive definite function $\psi: \mathfrak{S} \rightarrow \mathbb{C}$,
- (ii) $\varphi(s) = \int_{\mathfrak{X}_{\mathfrak{S}}} \hat{s} d\mu$ for all $s \in T$ with some positive measure μ on $\mathbf{M}_{\mathfrak{S}}$,
- (iii) $\Lambda_{\varphi}(p) \geq 0$ for every $p \in \mathcal{P}_T(\mathfrak{X}_{\mathfrak{S}})$ such that $p(\chi) \geq 0$ for all $\chi \in \mathfrak{X}_{\mathfrak{S}}$,

⁵ It follows from our assumptions that $\mathfrak{X}_{\mathfrak{S}}$ is determining, which makes it legitimate to consider Λ_{Φ} .

- (iv) $\Lambda_{\varphi,Y}(p) \geq 0$ for every $p \in \mathcal{P}_T(Y)$ for which there exist finitely many functions $q_1, \dots, q_n \in \mathcal{P}(Y)$ such that $p(\chi) = \sum_{j=1}^n |q_j(\chi)|^2$, $\chi \in Y$.

PROOF. Note first that the functional Λ_φ is completely positive if and only if (iii) holds. Indeed, if (iii) holds and $w = [w_{i,j}]_{i,j=1}^m \in M_+^m(\mathcal{P}_T(\mathfrak{X}_\Theta))$, then for every $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$, the function $p_\lambda \stackrel{\text{def}}{=} \sum_{i,j=1}^m \lambda_i \overline{\lambda_j} w_{i,j} \in \mathcal{P}_T(\mathfrak{X}_\Theta)$ is nonnegative on \mathfrak{X}_Θ , and consequently

$$\sum_{i,j=1}^m \lambda_i \overline{\lambda_j} \Lambda_\varphi(w_{i,j}) = \Lambda_\varphi(p_\lambda) \geq 0.$$

This means that $[\Lambda_\varphi(w_{i,j})]_{i,j=1}^m \geq 0$. The reverse implication is obvious.

The next observation is that the functional $\Lambda_{\varphi,Y}$ is completely f-positive if and only if (iv) holds. Indeed, if (iv) holds and $w = [\langle p_k, p_l \rangle]_{k,l=1}^m \in M_f^m(\mathcal{P}_T(Y))$ with some $p_1, \dots, p_m \in \mathcal{P}(Y, \ell^2)$, then for every $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$, the function $q_\lambda \stackrel{\text{def}}{=} \sum_{i=1}^m \lambda_i p_i$ belongs to $\mathcal{P}(Y, \ell^2)$ whereas $\langle q_\lambda, q_\lambda \rangle$ is in $\mathcal{P}_T(Y)$. Hence, by Lemma 6, we have

$$\sum_{i,j=1}^m \lambda_i \overline{\lambda_j} \Lambda_{\varphi,Y}(\langle p_i, p_j \rangle) = \Lambda_{\varphi,Y}(\langle q_\lambda, q_\lambda \rangle) \geq 0,$$

which means that $[\Lambda_{\varphi,Y}(\langle p_i, p_j \rangle)]_{i,j=1}^m \geq 0$. The reverse implication is plain.

The above enables us to adapt the proof of Theorem 14 to the present context. \square

APPLICATIONS

5. The truncated complex moment problem. The $*$ -semigroup \mathfrak{N}_+ we intend to investigate comes from [55]. It plays a crucial role in the complex moment problem. The initial part of this section is devoted to a description of $\mathcal{P}_T(Y)$ in this particular case.

Denote by \mathfrak{N}_+ the $*$ -semigroup $(\{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m + n \geq 0\}, +, *)$ with coordinatewise defined addition as semigroup operation and involution $(m, n)^* = (n, m)$. Owing to [55, Remark 7], the dual $*$ -semigroup $\mathfrak{X}_{\mathfrak{N}_+}$ can be identified algebraically with the $*$ -subsemigroup $(\Omega \cup (\{0\} \times \mathbb{T}), \cdot, *)$ of the product $*$ -semigroup $(\mathbb{C} \times \mathbb{T}, \cdot, *)$ (with coordinatewise defined multiplication as semigroup operation and involution $(z, w)^* = (\bar{z}, \bar{w})$), where

$$\Omega \stackrel{\text{def}}{=} \{(z, z \bar{z}^{-1}) : z \in \mathbb{C}_*\}.$$

Under this identification, $\widehat{(m, n)}$ is given by

$$(18) \quad \widehat{(m, n)}(z, w) = \chi_{z,w}(m, n) = \begin{cases} z^m \bar{z}^n & \text{if } z \neq 0, \\ 0 & \text{if } z = 0 \text{ and } m + n > 0, \\ w^m & \text{if } z = 0 \text{ and } m + n = 0, \end{cases}$$

for all $(z, w) \in \Omega \cup (\{0\} \times \mathbb{T})$ and $(m, n) \in \mathfrak{N}_+$. Given a subset Z of \mathbb{C}_* , we write

$$(19) \quad Y_Z = \{(z, z \bar{z}^{-1}) : z \in Z\} \subset \Omega.$$

By the above identification, we see that if T is a nonempty subset of \mathfrak{N}_+ , then $\mathcal{P}_T(Y_Z)$ may be regarded as the set of all rational functions p on Z of the form

$$(20) \quad p(z, \bar{z}) = \sum_{(m,n) \in T} a_{m,n} z^m \bar{z}^n, \quad z \in Z,$$

where $\{a_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ is a finite system.

For a subset T of $\mathbb{Z} \times \mathbb{Z}$, we denote by $\mathbb{C}_T(z, \bar{z})$ the set of all rational functions p of the form (20) with $Z = \mathbb{C}_*$ ($\{a_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ is a finite system). If $T \subset \mathfrak{N}$, where \mathfrak{N} is as in Example 3, then $\mathbb{C}_T(z, \bar{z})$ coincides with the set $\mathbb{C}_T[z, \bar{z}]$ of all complex polynomials in z and \bar{z} whose coefficients vanish for all indices in $\mathfrak{N} \setminus T$. If $T = \mathfrak{N}$, then we abbreviate $\mathbb{C}_T[z, \bar{z}]$ to $\mathbb{C}[z, \bar{z}]$. It is worth noticing that members of $\mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$ are functions, which are bounded on every punctured disc $\{z \in \mathbb{C}: 0 < |z| \leq R\}$, $R > 0$. This is due to the fact that all terms $z^m \bar{z}^n$ with $m + n \geq 0$ share this property. As a consequence, the set $\mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$ is essentially smaller than $\mathbb{C}_{\mathbb{Z} \times \mathbb{Z}}(z, \bar{z})$. In fact, members of $\mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$ can be characterized by their boundedness on the fixed punctured disc $\{z \in \mathbb{C}: 0 < |z| \leq 1\}$.

PROPOSITION 16. *If $p \in \mathbb{C}_{\mathbb{Z} \times \mathbb{Z}}(z, \bar{z})$, then the following conditions are equivalent*

- (i) $p \in \mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$,
- (ii) $\sup\{|p(z, \bar{z})|: z \in \mathbb{C}, 0 < |z| \leq 1\} < \infty$.

PROOF. Since the implication (i) \Rightarrow (ii) has been clarified above, we can focus on the implication (ii) \Rightarrow (i). Assume that (ii) holds and p is as in (20) with $T = \mathbb{Z} \times \mathbb{Z}$ and $Z = \mathbb{C}_*$. We may write the rational function p in the form $p = p_0 + \sum_{j=1}^k p_j$, where k is a positive integer, $p_0 \in \mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$ and

$$(21) \quad p_j(z, \bar{z}) = \sum_{\substack{m,n \in \mathbb{Z} \\ m+n=-j}} a_{m,n} z^m \bar{z}^n = |z|^{-j} \sum_{\substack{m,n \in \mathbb{Z} \\ m+n=-j}} a_{m,n} \left[\frac{z}{|z|} \right]^m \left[\frac{\bar{z}}{|z|} \right]^n$$

for $z \in \mathbb{C}_*$ and $j = 1, \dots, k$. Then

$$|z|^{k-1} p_k(z, \bar{z}) = |z|^{k-1} \left(p(z, \bar{z}) - p_0(z, \bar{z}) - \sum_{j=1}^{k-1} p_j(z, \bar{z}) \right), \quad z \in \mathbb{C}_*,$$

which, together with (ii) and (21), implies that the function $z \mapsto |z|^{k-1} p_k(z, \bar{z})$ is bounded on $\{z \in \mathbb{C}: 0 < |z| \leq 1\}$. Substituting $z = re^{i\theta}$ with real θ and $r > 0$, and using (21), we deduce that $p_k(z, \bar{z}) = 0$ for all $z \in \mathbb{C}_*$. Repeating this argument, we see that $p_1(z, \bar{z}) = \dots = p_k(z, \bar{z}) = 0$ for all $z \in \mathbb{C}_*$. By Lemma 5, this completes the proof. \square

Given $T \subset \mathbb{Z} \times \mathbb{Z}$, we say that a subset Z of \mathbb{C}_* is *determining* for $\mathbb{C}_T(z, \bar{z})$ (or shorter: $\mathbb{C}_T(z, \bar{z})$ -*determining*) if every rational function $p \in \mathbb{C}_T(z, \bar{z})$ vanishing on Z (i.e. $p(z, \bar{z}) = 0$ for all $z \in Z$) vanishes on \mathbb{C}_* . By Lemma 5, the set Z is $\mathbb{C}_T(z, \bar{z})$ -determining if and only if the system of functions $z \mapsto z^m \bar{z}^n$, $(m, n) \in T$, is linearly independent in \mathbb{C}^Z . If $T \subset \mathbb{Z}_+ \times \mathbb{Z}_+$, then we allow 0 to be a member of a determining set. Clearly, if $T \subset \mathbb{Z} \times \mathbb{Z}$ and $Z \subset \mathbb{C}_*$ is a determining set for $\mathbb{C}_{\mathbb{Z} \times \mathbb{Z}}(z, \bar{z})$, then Z is determining for $\mathbb{C}_T(z, \bar{z})$. The reverse is not true, e.g. for $T = \{(n, n): n \in \mathbb{Z}_+\}$ and $Z = \{z \in \mathbb{C}_*: z = \bar{z}\}$. Lemma 5 provides examples of $\mathbb{C}_{\mathbb{Z} \times \mathbb{Z}}(z, \bar{z})$ -determining sets. A particular class of them is indicated below.

The following simple fact is crucial for further investigations. It enables us to deal with the complex plane instead of a larger and less handy dual \ast -semigroup $\mathfrak{X}_{\mathfrak{N}_+}$.

PROOF. Apply Lemma 5 and the description (18) of $\widehat{\mathfrak{N}}_+$. \square

LEMMA 19. *If Z is a subset of \mathbb{C} , then the following conditions are equivalent*

- PROOF. (i) \Rightarrow (iii) Suppose that $Z_0 = \{\lambda_1, \dots, \lambda_k\}$ and take $p \in \mathbb{C}_{z \times \bar{z}}(z, \bar{z})$ such that $p(z, \bar{z}) = 0$ for all $z \in Z \setminus (Z_0 \cup \{0\})$. Then for a sufficiently large positive integer N the rational function $q(z, \bar{z}) \stackrel{\text{def}}{=} z^N \bar{z}^N (z - \lambda_1) \dots (z - \lambda_k) p(z, \bar{z})$ is a polynomial in z and \bar{z} such that $q(z, \bar{z}) = 0$ for all $z \in Z$. By (i), $q(z, \bar{z}) = 0$ for all $z \in \mathbb{C}$, and so $p(z, \bar{z}) = 0$ for all $z \in \mathbb{C} \setminus (Z_0 \cup \{0\})$. As a consequence, $p(z, \bar{z}) = 0$ for all $z \in \mathbb{C}_*$.

A sequence $\{c_{m,n}\}_{m,n=0}^\infty \subset \mathbb{C}$ is called a *complex moment sequence* if there exists a positive Borel measure μ on \mathbb{C} such that

Such a measure μ is called a *representing measure* of $\{c_{m,n}\}_{m,n=0}^\infty$; it is by no means unique. If a representing measure μ is unique, then $\{c_{m,n}\}_{m,n=0}^\infty$ is called a *determinate* complex moment sequence. For this and related questions we refer the reader to [48], [23] and [55].

The following result is an extension of Theorem 1 of [55] as well as of the complex version of the Riesz-Haviland theorem (see Theorem B). The first of the two theorems can be seen as the equivalence (i) \Leftrightarrow (iii) below with $T = \mathbb{Z}_+ \times \mathbb{Z}_+$, while the other as the equivalence (i) \Leftrightarrow (iv) with the same T .

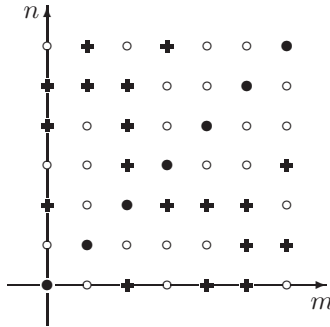


Figure 1. T appearing in Theorem 20 – an example:

• - obligatory data, + - additional data, o - missing data.

THEOREM 20. *Let T be a symmetric subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$ (i.e. $(n, m) \in T$ for all $(m, n) \in T$) containing the diagonal $\{(n, n) : n \in \mathbb{Z}_+\}$, and let $Z \subset \mathbb{C}$ be a determining set for $\mathbb{C}[z, \bar{z}]$. Then for any system of complex numbers $\{c_{m,n}\}_{(m,n) \in T}$, the following conditions are equivalent:*

(i) *there exists a positive Borel measure μ on \mathbb{C} such that*

$$c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mu(dz), \quad (m, n) \in T,$$

- (ii) *there exists a complex moment sequence $\{\tilde{c}_{m,n}\}_{m,n=0}^\infty$ such that $\tilde{c}_{m,n} = c_{m,n}$ for all $(m, n) \in T$,*
 (iii) *there exists⁶ $\{\tilde{c}_{m,n}\}_{m+n \geq 0} \subset \mathbb{C}$ such that $\tilde{c}_{m,n} = c_{m,n}$ for all $(m, n) \in T$, and $\sum_{m+n \geq 0} \tilde{c}_{m+q, n+p} \lambda_{m,n} \bar{\lambda}_{p,q} \geq 0$ for all finite systems $\{\lambda_{m,n}\}_{m+n \geq 0} \subset \mathbb{C}$,*
 (iv) *$\sum_{(m,n) \in T} p_{m,n} c_{m,n} \geq 0$ for every finite system $\{p_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ such that $\sum_{(m,n) \in T} p_{m,n} z^m \bar{z}^n \geq 0$ for all $z \in \mathbb{C}$,*
 (v) *$\sum_{(m,n) \in T} p_{m,n} c_{m,n} \geq 0$ for every finite system $\{p_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ for which there exist finitely many rational functions $q_1, \dots, q_k \in \mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$ such that $\sum_{(m,n) \in T} p_{m,n} z^m \bar{z}^n = \sum_{j=1}^k |q_j(z, \bar{z})|^2$ for all $z \in \mathbb{C}$, $z \neq 0$,*
 (vi) *$\sum_{(m,n) \in T} p_{m,n} c_{m,n} \geq 0$ for every finite system $\{p_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ for which there exist finitely many rational functions $q_1, \dots, q_k \in \mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$ such that $\sum_{(m,n) \in T} p_{m,n} z^m \bar{z}^n = \sum_{j=1}^k |q_j(z, \bar{z})|^2$ for all $z \in Z$, $z \neq 0$.*

PROOF. (i) \Rightarrow (ii) Since $\{(n, n) : n \in \mathbb{Z}_+\} \subset T$, we see that complex polynomials in z and \bar{z} are summable with respect to μ . Hence, we can define $\tilde{c}_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mu(dz)$ for $m, n \in \mathbb{Z}_+$, which is the desired complex moment sequence.

(ii) \Leftrightarrow (iii) This can be deduced from [55, Theorem 1].

Implications (ii) \Rightarrow (i), (i) \Rightarrow (iv) and (iv) \Rightarrow (v) are evident.

(v) \Rightarrow (vi) By Lemma 19, $Z \setminus \{0\}$ is a determining set for $\mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$. Since both sides of the equality in (vi) are members of $\mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$, and they coincide on $Z \setminus \{0\}$, we deduce that they are equal on \mathbb{C}_* .

(vi) \Rightarrow (iii) In view of Lemmata 18 and 19, $Y_{Z \setminus \{0\}}$ is a determining set for $\mathcal{P}(\mathfrak{X}_{\mathfrak{N}_+})$. Since \mathfrak{N}_+ is a semiperfect $*$ -semigroup (this can be deduced from⁷ [55, Remark 7]), we can apply implication (iv) \Rightarrow (i) of Theorem 15 to $\mathfrak{S} = \mathfrak{N}_+$, $Y = Y_{Z \setminus \{0\}}$ and $\varphi(m, n) = c_{m,n}$. This completes the proof. \square

Let us point out the difference between the usual meaning of the truncated (complex) moment problem (where a finite system of complex numbers is given) and that appearing in Theorem 20. One of our assumptions requires for a system of complex numbers $\{c_{m,n}\}_{(m,n) \in T}$, which is the given data, to include all the diagonal entries $c_{m,m}$, $m = 0, 1, 2, \dots$. This enabled us to show that the most direct analogue of the complex version of the Riesz-Haviland theorem for the truncated moment

⁶ Notation $m + n \geq 0$ has to be understood as $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ and $m + n \geq 0$.

⁷ Indeed, in view of the discussion preceding Lemma 12, we see that our meaning of semiperfectness is wider than that of [55, Remark 7]. The interested reader can verify that for finitely generated unital commutative $*$ -semigroups, like \mathfrak{N}_+ , both these notions are equivalent.

problem in our meaning is true. In a recent paper [21, Example 2.1], Curto and Fialkow have shown that this is not the case for the truncated moment problem in the usual sense. Instead, they have found an analogue of the Riesz-Haviland theorem, which requires extending appropriate linear functionals on polynomials of degree limited by $2n$ to positive linear functionals on polynomials of degree limited by $2n + 2$ (cf. [21, Theorem 2.2.]).

5.1. The case of $(\mathfrak{N}_+)_h \not\subset T$. In this subsection we show that the assumption that T contains the diagonal $\{(n, n) : n \in \mathbb{Z}_+\}$ cannot be dismissed without destroying the equivalences (i) \Leftrightarrow (ii) as well as (i) \Leftrightarrow (iv) of Theorem 20.

EXAMPLE 21. Clearly, the implication (ii) \Rightarrow (i) of Theorem 20 holds for an arbitrary subset T of $\mathbb{Z}_+ \times \mathbb{Z}_+$. The reverse implication does not hold in general even if T contains all but a finite number of the diagonal elements (hence all but a finite number of the moments exist). This can be shown for any subset T of $\mathbb{Z}_+ \times \mathbb{Z}_+$ such that $(0, 0) \notin T$ and

$$\{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : m, n \geq k\} \subset T$$

with some integer $k \geq 1$. For this, define the positive Borel measure μ on \mathbb{C} via $d\mu(z) = |z|^{-2}\eta(z)dV(z)$, where V stands for the planar Lebesgue measure and η is the characteristic function of the disc $\Delta \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| \leq 1\}$. Since $(0, 0) \notin T$, the system $c_{m,n} \stackrel{\text{def}}{=} \int_{\mathbb{C}} z^m \bar{z}^n d\mu(z)$, $(m, n) \in T$, is well defined and fulfils the condition (i) of Theorem 20. Suppose that, contrary to our claim, it satisfies the condition (ii) of Theorem 20. Then there exists a finite positive Borel measure ν on \mathbb{C} such that $c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n d\nu(z)$ for all $(m, n) \in T$. Hence, we have

$$\int_{\Delta} z^m \bar{z}^n |z|^{2k} dV(z) = \int_{\mathbb{C}} z^m \bar{z}^n |z|^{2k+2} d\nu(z), \quad m, n \in \mathbb{Z}_+.$$

Since the left-hand side represents a determinate complex moment sequence indexed by (m, n) (as it has a compactly supported representing measure), we see that

$$|z|^{2k}\eta(z)dV(z) = |z|^{2k+2}d\nu(z).$$

It follows that $\mu(\mathbb{C}_*) = \nu(\mathbb{C}_*) < \infty$, which is a contradiction because μ is not a finite measure. This proves our claim.

In view of [55, Theorem 1], the extension procedure required in the condition (ii) of Theorem 20 can be realized in two steps: first we have to extend the system $\{c_{m,n}\}_{(m,n) \in T}$ to a positive definite system over the $*$ -semigroup \mathfrak{N} , and then to a positive definite system over \mathfrak{N}_+ . The first step can be done in a more explicit way for $*$ -ideals T of \mathfrak{N} (i.e. $T = T^*$ and $T + \mathfrak{N} \subset T$) as in [61]; see also [37, 58, 50] for earlier attempts in this direction. However, the $*$ -ideal technique is not applicable in the other step when extending positive definite functions from \mathfrak{N} to \mathfrak{N}_+ . This situation requires methods invented in [55].

We next consider the case of the equivalence (i) \Leftrightarrow (iv) of Theorem 20.

EXAMPLE 22. It is evident that the implication (i) \Rightarrow (iv) of Theorem 20 holds for sets T not necessarily containing the diagonal $\{(m, m) : m \in \mathbb{Z}_+\}$. The reverse implication does not hold in general, which can be shown for all sets T such that

$$(22) \quad \{(k, k), (l, l)\} \subset T \subset \{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : m, n \geq k\},$$

where k, l are integers such that $1 \leq k < l$. Indeed, consider the system $\mathbf{c} = \{\delta_{0, m+n-2k}\}_{(m,n) \in T}$, where $\delta_{i,j}$ is the Kronecker delta function. It can be readily

checked that \mathbf{c} satisfies the condition (iv) of Theorem 20 (see the limit formula in the proof of (b) \Rightarrow (a) of Proposition 23). Suppose that, contrary to our claim, it satisfies the condition (i) of Theorem 20 with some positive Borel measure μ on \mathbb{C} . Then $1 = \delta_{0,0} = \int_{\mathbb{C}} |z|^{2k} d\mu(z)$ and $0 = \delta_{0,2(l-k)} = \int_{\mathbb{C}} |z|^{2l} d\mu(z)$. The latter equality implies that μ is supported in $\{0\}$, which contradicts the former (because $k \geq 1$). In the extremal case T may consist of only two elements, which is the smallest possible number required for the above argument because for one point sets T of the form $\{(k, k)\}$ with $k \in \mathbb{Z}_+$ the equivalence (i) \Leftrightarrow (iv) of Theorem 20 is valid. Note also that if $T = \{(0, 0), (l, l)\}$ with $l \geq 1$, then the system $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ given by $c_{0,0} = 0$ and $c_{l,l} = 1$ satisfies the condition (iv) of Theorem 20 but not (i).

The ensuing proposition shows that for a subclass of sets T satisfying (22) the condition (iv) of Theorem 20 leads to a representation similar (but not equivalent if $k \geq 1$) to that in (i) of Theorem 20. Proposition 23 is somehow in the flavour of [62] where backward extensions of moment sequences are considered.

PROPOSITION 23. *Let T be a symmetric subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$ such that*

$$(23) \quad \{(m, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : m \geq k\} \subset T \subset \{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : m, n \geq k\}$$

with some integer $k \geq 0$, and let $\{c_{m,n}\}_{(m,n) \in T}$ be a system of complex numbers. Then the following conditions are equivalent:

- (a) $\sum_{(m,n) \in T} p_{m,n} c_{m,n} \geq 0$ for every finite system $\{p_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ such that $\sum_{(m,n) \in T} p_{m,n} z^m \bar{z}^n \geq 0$ for all $z \in \mathbb{C}$,
- (b) *there exist a positive Borel measure μ on \mathbb{C} and a real $a \geq 0$ such that $\mu(\{0\}) = 0$ and*

$$(24) \quad c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n d\mu(z) + a \delta_{0,m+n-2k}, \quad (m, n) \in T.$$

In particular, $\int_{\mathbb{C}} |z|^{2k} d\mu(z) < \infty$ for the measure μ appearing in (b).

PROOF. Let $\Lambda: \mathbb{C}_T[z, \bar{z}] \rightarrow \mathbb{C}$ be a linear functional determined by $\Lambda(z^m \bar{z}^n) = c_{m,n}$ for all $(m, n) \in T$. A polynomial $p \in \mathbb{C}[z, \bar{z}]$ is called *nonnegative* if $p(z, \bar{z}) \geq 0$ for all $z \in \mathbb{C}$.

(a) \Rightarrow (b) Note that the set $T_k \stackrel{\text{def}}{=} \{(m-k, n-k) : (m, n) \in T\}$ satisfies the assumptions of Theorem 20. Since $\Lambda(z^k \bar{z}^k q(z, \bar{z})) \geq 0$ for all nonnegative polynomials $q \in \mathbb{C}_{T_k}[z, \bar{z}]$, we infer from the implication (iv) \Rightarrow (i) of Theorem 20 that there exists a finite positive Borel measure ν on \mathbb{C} such that $\Lambda(z^k \bar{z}^k q(z, \bar{z})) = \int_{\mathbb{C}} q(z, \bar{z}) d\nu(z)$ for all $q \in \mathbb{C}_{T_k}[z, \bar{z}]$. Hence

$$\begin{aligned} c_{m,n} &= \Lambda(z^k \bar{z}^k (z^{m-k} \bar{z}^{n-k})) = \int_{\mathbb{C}} z^{m-k} \bar{z}^{n-k} d\nu(z) \\ &= \int_{\mathbb{C}_*} z^m \bar{z}^n d\mu(z) + \nu(\{0\}) \delta_{0,m+n-2k}, \quad (m, n) \in T, \end{aligned}$$

where μ is the positive Borel measure on \mathbb{C} given by

$$\mu(\sigma) = \int_{\sigma \setminus \{0\}} \frac{1}{|z|^{2k}} d\nu(z), \quad \sigma - \text{a Borel subset of } \mathbb{C}.$$

This μ and $a \stackrel{\text{def}}{=} \nu(\{0\})$ satisfy (b) as well as the “in particular” part of the conclusion.

(b) \Rightarrow (a) Pick a nonnegative polynomial $p \in \mathbb{C}_T[z, \bar{z}]$ and denote its (k, k) -coefficient by $p_{k,k}$. Then $\Lambda(p) = \int_{\mathbb{C}} p(z, \bar{z}) d\mu(z) + ap_{k,k}$ which is nonnegative because p is nonnegative and $p_{k,k} = \lim_{z \rightarrow 0} |z|^{-2k} p(z, \bar{z})$. \square

Clearly, the implication (b) \Rightarrow (a) of Proposition 23 holds with the same proof if it is only assumed that T is a (not necessarily symmetric) subset of the set $\{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : m, n \geq k\}$ with some integer $k \geq 0$. Note also that for every integer $k \geq 1$ and for every symmetric subset T of $\mathbb{Z}_+ \times \mathbb{Z}_+$ satisfying (23), we may find a system $\{c_{m,n}\}_{(m,n) \in T}$ fulfilling the condition (b) of Proposition 23 with μ such that $\int_{\mathbb{C}} |z|^{2k} d\mu(z) < \infty$ and $\int_{\mathbb{C}} |z|^{2l} d\mu(z) = \infty$ for all $l = 0, \dots, k-1$. Indeed, such a system can be produced with the help of the formula (24) with $a = 0$ and μ given by $d\mu(z) = |z|^{-2k} \eta(z) dV(z)$, where η and V are as in Example 21.

5.2. The lack of symmetry. The phenomenon described below is of general nature and as such occurs in other instances, like the truncated multidimensional trigonometric moment problem (cf. Theorem 34) and the truncated two-sided complex moment problem (cf. Theorem 40). Let us discuss it in the case of the truncated complex moment problem, which is related to the $*$ -semigroup \mathfrak{N}_+ , leaving the other cases for the reader.

Consider a not necessarily symmetric set T such that

$$(25) \quad (\mathfrak{N}_+)_h \subset T \subset \mathfrak{N}$$

and look at what happens to the equivalence of the conditions (i)–(vi) of Theorem 20 (the other assumptions of Theorem 20 being still in force). First of all, we see the two natural candidates for replacing T by a symmetric set: $T \cup T^*$ and $T \cap T^*$, both satisfying (25). As shown below, the set $T \cup T^*$ plays an essential role in conditions (i)–(iii) while $T \cap T^*$ does so in (iv)–(vi); because these two sets for very nonsymmetric T 's may differ in the extreme the aforesaid feature seems to be worthy of taking a closer look at.

Call a system $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ *symmetrizable* if

$$c_{m,n} = \overline{c_{n,m}}, \quad (m, n) \in T \cap T^*.$$

If $\{c_{m,n}\}_{(m,n) \in T}$ is symmetrizable, then its *symmetrization* $\{c_{m,n}^\sharp\}_{(m,n) \in T \cup T^*}$:

$$c_{m,n}^\sharp = \begin{cases} c_{m,n}, & (m, n) \in T, \\ \overline{c_{n,m}}, & (m, n) \in T^*, \end{cases}$$

is well defined. One can verify that if a system $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ satisfies (i) (respectively: (ii), (iii)) on T , then it is symmetrizable and its symmetrization satisfies (i) (respectively: (ii), (iii)) on $T \cup T^*$, and vice versa. Theorem 20 implies that, via the symmetrization procedure, for *any* set T obeying (25) the conditions (i)–(iii) are equivalent on T .

Regarding the conditions (iv)–(vi), their prospective equivalence needs to be justified in a different way. Namely, a system $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ fulfils (iv) (respectively: (v), (vi)) on T if and only if the restricted system $\{c_{m,n}\}_{(m,n) \in T \cap T^*}$ fulfils (iv) (respectively: (v), (vi)) on $T \cap T^*$ (hence $c_{m,n}$'s over $T \setminus T^*$ are irrelevant). Indeed, this can be deduced from the fact that a real-valued polynomial $p \in \mathbb{C}[z, \bar{z}]$ belongs to $\mathbb{C}_T[z, \bar{z}]$ if and only if it belongs to $\mathbb{C}_{T \cap T^*}[z, \bar{z}]$ (hint: conjugate the polynomial p and deduce that $p_{m,n} = \overline{p_{n,m}}$, where $p_{m,n}$ are the coefficients of p). Hence,

by Theorem 20, for *any* set T satisfying (25) the conditions (iv)–(vi) are equivalent on T . If this happens, then the system $\{c_{m,n}\}_{(m,n) \in T}$ is symmetrizable.

Since, in fact, the conditions (i)–(iii) concern the extension of the system $\{c_{m,n}\}_{(m,n) \in T}$ to $T \cup T^*$, while (iv)–(vi) deal with its restriction to $T \cap T^*$, it is to be expected that they cannot be altogether equivalent for arbitrary T . Indeed, consider any nonsymmetric set T satisfying (25) and take $(k, l) \in T \setminus T^*$. Suppose that a system $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ fulfils the conditions (iv)–(vi) on T (e.g. any restriction to T of a complex moment sequence does). Define the new system

$$\tilde{c}_{m,n} = \begin{cases} c_{m,n}, & (m, n) \in T \setminus \{(k, l)\}, \\ \sqrt{c_{k,k}c_{l,l} + 1}, & (m, n) = (k, l). \end{cases}$$

Owing to the above discussion the so defined system satisfies the conditions (iv)–(vi), but fails to satisfy any of the conditions (i)–(iii), because otherwise by the Cauchy-Schwarz inequality we would have $|\tilde{c}_{k,l}|^2 \leq \tilde{c}_{k,k}\tilde{c}_{l,l} = c_{k,k}c_{l,l}$, a contradiction. This means that Theorem 20 is not true as long as T satisfying (25) is not symmetric. In other words, symmetricity of T is a necessary condition for Theorem 20 to hold.

However, it becomes now clear that for an *arbitrary* set T satisfying (25) a seemingly more general version of Theorem 20 can be considered. Putting it precisely, for a symmetrizable system $\{c_{m,n}\}_{(m,n) \in T}$ all the conditions (i)–(vi) remain equivalent if in the conditions (iv)–(vi) the system $\{c_{m,n}\}_{(m,n) \in T}$ is replaced by its symmetrization $\{c_{m,n}^\sharp\}_{(m,n) \in T \cup T^*}$.

5.3. Sum-square representation. We say that a locally convex topology τ on the linear space $\mathbb{C}[z, \bar{z}]$ is *evaluable* if the set

$$\{\lambda \in \mathbb{C}: \text{the evaluation } E_\lambda: \mathbb{C}[z, \bar{z}] \ni p \mapsto p(\lambda, \bar{\lambda}) \in \mathbb{C} \text{ is } \tau\text{-continuous}\}$$

is dense in \mathbb{C} . The class of such topologies is rich. In particular, it contains every locally convex topology on $\mathbb{C}[z, \bar{z}]$ generated by the family $\{E_\lambda: \lambda \in Z\}$, where Z is a dense subset of \mathbb{C} . This fact, the linear independence of $\{E_\lambda: \lambda \in \mathbb{C}\}$ and [46, Theorem 3.10] imply that there exist two evaluable topologies such that the only linear functional on $\mathbb{C}[z, \bar{z}]$ continuous with respect to each of them is the zero functional.

One can deduce from Artin's solution of the 17th Hilbert problem (cf. [3] or [14, Theorem 6.1.1]; see also [43, 42] for the case of positive homogeneous polynomials) that for every nonnegative polynomial $p \in \mathbb{C}[z, \bar{z}]$ (i.e. $p(z, \bar{z}) \geq 0$ for all $z \in \mathbb{C}$), there exist finitely many rational functions q_1, \dots, q_n in two complex variables such that $p(z, \bar{z}) = \sum_{j=1}^n |q_j(z, \bar{z})|^2$ for all $z \in \mathbb{C}$ except singularities of the right-hand side of the equality. The question arises whether general rational functions in the above representation of p can be replaced by specific ones from $\mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$. Though we are unable to answer this question in full generality, we can do it in the case in which the nonnegativity of p and its sum-square representation is considered on a closed proper subset Z of \mathbb{C} which is determining for $\mathbb{C}[z, \bar{z}]$. For convenience we denote by $\Sigma^2(Z)$ the set of all polynomials $q \in \mathbb{C}[z, \bar{z}]$ for which there exist finitely many rational functions $q_1, \dots, q_n \in \mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$ such that

$$(26) \quad q(z, \bar{z}) = \sum_{j=1}^n |q_j(z, \bar{z})|^2, \quad z \in Z \setminus \{0\}.$$

Perhaps this is the right place to mention that if $q \in \mathbb{C}[z, \bar{z}]$ is of the form (26) with $Z = \mathbb{C}$ and $q_1, \dots, q_n \in \mathbb{C}_{\mathbb{Z} \times \mathbb{Z}}(z, \bar{z})$, then by Proposition 16 the rational functions q_1, \dots, q_n must belong to $\mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$.

PROPOSITION 24. *Let Z be a closed proper subset of \mathbb{C} which is determining for $\mathbb{C}[z, \bar{z}]$ and let τ be an evaluable locally convex topology on $\mathbb{C}[z, \bar{z}]$. Then there exists a polynomial $p \in \mathbb{C}[z, \bar{z}]$ which is nonnegative on Z and which does not belong to the τ -closure of $\Sigma^2(Z)$. In particular, p is not in $\Sigma^2(Z)$.*

PROOF. Denote by $\mathcal{P}^+(Z)$ the set of all polynomials $q \in \mathbb{C}[z, \bar{z}]$ which are nonnegative on Z (i.e. $q(z, \bar{z}) \geq 0$ for all $z \in Z$) and write $\overline{\Sigma^2(Z)}^\tau$ for the τ -closure of $\Sigma^2(Z)$. It follows from our assumptions that there exists $\lambda \in \mathbb{C} \setminus Z$ such that the evaluation E_λ is τ -continuous. By the determining property of $Z \setminus \{0\}$ (cf. Lemma 19), we have $\Sigma^2(Z) = \Sigma^2(\mathbb{C})$. Hence

$$(27) \quad E_\lambda(q) \geq 0, \quad q \in \overline{\Sigma^2(Z)}^\tau.$$

Since $\lambda \notin Z$, there exists real $\varepsilon > 0$ such that $\{z \in \mathbb{C} : |z - \lambda| \leq \varepsilon\} \subset \mathbb{C} \setminus Z$. It is then clear that the polynomial $p_\varepsilon(z, \bar{z}) \stackrel{\text{def}}{=} |z - \lambda|^2 - \varepsilon^2$ belongs to $\mathcal{P}^+(Z)$. Note that $p_\varepsilon \notin \overline{\Sigma^2(Z)}^\tau$. Indeed, otherwise (27) implies that $E_\lambda(p_\varepsilon) \geq 0$, which contradicts $E_\lambda(p_\varepsilon) = -\varepsilon^2$. \square

The proof of Proposition 24 remains unchanged if we assume only that τ is a locally convex topology on $\mathbb{C}[z, \bar{z}]$ for which there exists $\lambda \in \mathbb{C} \setminus Z$ such that the evaluation E_λ is τ -continuous.

5.4. Determining sets versus supports of representing measures.

Notice that no determining set is mentioned in the condition (v) of Theorem 20. On the other hand, this condition remains equivalent to the variety of conditions obtained from (vi) by taking all $\mathbb{C}[z, \bar{z}]$ -determining subsets Z of \mathbb{C} . The same observation refers to the mutual relationship between (iv) and (vi). Our intension now is to see what happens if in (iv) the phrase “ $z \in \mathbb{C}$ ” is replaced by “ $z \in Z$ ”; denote such a modified condition by $(iv)_Z$ (the same operation applied to (v) leads to (vi)). Evidently, if a system $\{c_{m,n}\}_{(m,n) \in T}$ satisfies $(iv)_Z$, then it also satisfies (iv). We will discuss the following two questions:

- 1° given a symmetric set T obeying (25) and a nonzero system $\{c_{m,n}\}_{(m,n) \in T}$ of complex numbers, is (iv) equivalent to $(iv)_Z$ for any $\mathbb{C}[z, \bar{z}]$ -determining set $Z \subset \mathbb{C}$?
- 2° given a symmetric set T obeying (25) and a closed⁸ proper subset Z of \mathbb{C} , is (iv) equivalent to $(iv)_Z$ for any system $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$?

We will use the notation

$$Y_r = \{|z|^2 : z \in Y\} \text{ and } \sqrt{Y_r} = \{|z| : z \in Y\} \text{ for any } Y \subset \mathbb{C}.$$

Observe that if Y is closed, then so are Y_r and $\sqrt{Y_r}$. In order to handle the questions just posed, we need the following lemma.

LEMMA 25. *If T obeys (25), Z is a subset of \mathbb{C} and $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ is a system satisfying $(iv)_Z$, then $\{c_{m,m}\}_{m=0}^\infty$ is a Stieltjes moment sequence which has a representing measure supported in $\sqrt{Z_r}$.*

⁸ Note that $(iv)_Z$ is equivalent to $(iv)_{\overline{Z}}$ for any $Z \subset \mathbb{C}$.

PROOF. If $p(x) = \sum_{j=0}^k p_j x^j \in \mathbb{C}[x]$ is nonnegative on Z_r , then $p(z\bar{z}) \in \mathbb{C}_T[z, \bar{z}]$ is nonnegative on Z , and consequently, by (iv) $_Z$, $\sum_{j=0}^k p_j c_{j,j} \geq 0$. Applying Theorem A with $\varkappa = 1$, we see that the Stieltjes moment sequence $\{c_{m,m}\}_{m=0}^\infty$ has a desired representing measure. \square

The answer to the question 1° is in the negative. Indeed, take a nonzero system $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ satisfying (iv) and suppose that, contrary to our claim, the system satisfies (iv) $_Z$ for all $\mathbb{C}[z, \bar{z}]$ -determining sets $Z \subset \mathbb{C}$. In particular, this is the case for $Z_1 = \{z \in \mathbb{C} : |z| \leq 1\}$ and $Z_2 = \{z \in \mathbb{C} : 2 \leq |z| \leq 3\}$ (for their determining property see Lemma 5). By Lemma 25, $\{c_{m,m}\}_{m=0}^\infty$ is a Stieltjes moment sequence having two representing measures supported in $[0, 1]$ and $[4, 9]$, respectively. Since each Hamburger moment sequence with a compactly supported representing measure is determinate (cf. [23]), we deduce that the support of the unique representing measure of $\{c_{m,m}\}_{m=0}^\infty$ is empty. Therefore $c_{m,m} = 0$ for all $m \in \mathbb{Z}_+$. By Theorem 20 (i) and the Cauchy-Schwarz inequality, we have $|c_{m,n}|^2 \leq c_{m,m}c_{n,n} = 0$ for all $(m, n) \in T$, a contradiction.

Regarding the question 1° with $T = \mathfrak{N}$, it is possible to find a complex moment sequence $\{c_{m,n}\}_{m,n=0}^\infty$ which fulfils (iv) $_Z$ with uncountably many pairwise disjoint sets Z . To see this consider any indeterminate Hamburger moment sequence $\{a_n\}_{n=0}^\infty \subset \mathbb{R}$ and set $c_{m,n} = a_{m+n}$ for $m, n \in \mathbb{Z}_+$. It follows from [49, Theorem 4.11] that $\{a_n\}_{n=0}^\infty$ has a family \mathscr{W} (necessarily of cardinality continuum) of representing measures μ whose closed supports $\text{supp } \mu$ are infinite and pure point (i.e. with no cluster points), and form a partition of the real line. It is now easily seen that the closed sets $Z_\mu \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \Re z \in \text{supp } \mu\}$, $\mu \in \mathscr{W}$, are determining for $\mathbb{C}[z, \bar{z}]$ (see Lemma 5) and the sequence $\{c_{m,n}\}_{m,n=0}^\infty$ satisfies (iv) $_{Z_\mu}$ for every $\mu \in \mathscr{W}$. Note that the family $\{Z_\mu\}_{\mu \in \mathscr{W}}$ is a partition of \mathbb{C} . An example of an indeterminate Hamburger moment sequence $\{a_n\}_{n=0}^\infty$ with explicitly computed pure point supports of representing measures forming a partition of \mathbb{R} may be found in [19] (see also [18] for an explicit example of an indeterminate Stieltjes moment sequence with continuum of representing measures).

The answer to the question 2° depends essentially on the interplay between the sets T and Z . We do not demand that Z be $\mathbb{C}[z, \bar{z}]$ -determining, however, this can be guaranteed in all the examples presented below. We will first take a closer look at the extremal case $T = \mathfrak{N}$ (the other extremality $T = (\mathfrak{N}_+)_h$ is discussed below). Then any determinate nonzero complex moment sequence with the representing measure supported in $\mathbb{C} \setminus Z$ satisfies (iv), but not (iv) $_Z$, the latter being a consequence of Theorem B. Such a moment sequence always exists; e.g. it can be produced from any nonzero finite positive Borel measure on \mathbb{C} compactly supported in $\mathbb{C} \setminus Z$; for the determinacy of the so obtained moment sequence see [23]. Hence, in this particular case, the answer to the question 2° is in the negative. An alternative way to achieve this conclusion is by applying Proposition 28 below.

Another instance of the negative answer to 2° is when T and Z are as in 2° and $Z_r \subsetneq [0, \infty)$. For this we may consider a nonzero complex moment sequence $\{c_{m,n}\}_{m,n=0}^\infty$ with a representing measure compactly supported in the open set $\{\lambda \in \mathbb{C} : |\lambda|^2 \notin Z_r\}$. By the measure transport theorem (or Lemma 25) the Stieltjes moment sequence $\{c_{m,m}\}_{m=0}^\infty$ has a representing measure compactly supported in $[0, \infty) \setminus Z_r$ and as such is determinate. It turns out that the system $\{c_{m,n}\}_{(m,n) \in T}$ satisfies (iv), but not (iv) $_Z$. Indeed, if it satisfied (iv) $_Z$, then by Lemma 25 the

moment sequence $\{c_{m,m}\}_{m=0}^\infty$ would have a representing measure supported in Z_r . Again we would deduce that the representing measure of $\{c_{m,m}\}_{m=0}^\infty$ is the zero measure and hence $c_{m,n} = 0$ for all $m, n \in \mathbb{Z}_+$, a contradiction.

However, the answer to the question 2° is in the affirmative when $T = (\mathfrak{N}_+)_\mathfrak{h}$ and $Z_r = [0, \infty)$. To see this it suffices to notice that for every $Y \subset \mathbb{C}$ the system $\{c_{m,n}\}_{(m,n) \in (\mathfrak{N}_+)_\mathfrak{h}}$ satisfies (iv)_Y if and only if the sequence $\{c_{m,m}\}_{m=0}^\infty$ satisfies the *Riesz-Haviland positivity condition* on Y_r , i.e. $\sum_{j=0}^k p_j c_{j,j} \geq 0$ for every polynomial $p(x) = \sum_{j=0}^k p_j x^j \in \mathbb{C}[x]$ which is nonnegative on Y_r . Since $Z_r = \mathbb{C}_r = [0, \infty)$, we get the desired conclusion.

We now provide more elaborate examples of T and Z for which the answer to the question 2° remains affirmative. Fix integers $l > k \geq 0$ and set

$$\mathcal{T}_{k,l} = (\mathfrak{N}_+)_\mathfrak{h} \cup \{(k,l), (l,k)\}.$$

Clearly, $\mathcal{T}_{k,l}$ is symmetric and fulfils (25). By Proposition 26 below, the answer to the question 2° is in the affirmative whenever $l - k$ is even, $T = \mathcal{T}_{k,l}$ and $Z = \mathcal{Z}\left(\frac{2\pi}{l-k}\right)$, where

$$\mathcal{Z}(\alpha) \stackrel{\text{def}}{=} \{\varrho e^{it} : t \in [0, \alpha], \varrho \geq 0\}, \quad \alpha \in [0, 2\pi];$$

note that due to Lemma 17 the set $\mathcal{Z}(\alpha)$ is $\mathbb{C}[z, \bar{z}]$ -determining for $\alpha > 0$. The case of $l - k$ being an arbitrary integer greater than or equal to 2 will be settled affirmatively in Proposition 27 below, however its proof making use of Theorem 20 is no longer elementary. What is more, while Proposition 26 is stated purely in terms of the system $\{c_{m,n}\}_{(m,n) \in T}$, this seems to be impossible in the case of Proposition 27 (apart from some restricted cases in which the square root can be approximated by polynomials in L^2 -norm with respect to a representing measure of $\{c_{m,m}\}_{m=0}^\infty$, e.g. when the representing measure is N-extremal, cf. [49]). According to footnote 8 and the equality $(\bar{Z})_r = \overline{Z_r}$, there is no loss of generality in assuming that Z is closed.

PROPOSITION 26. *Let $T = \mathcal{T}_{k,l}$ with $\varkappa \stackrel{\text{def}}{=} (l - k)/2$ being a positive integer and let Z be a closed $\mathbb{C}[z, \bar{z}]$ -determining subset of \mathbb{C} such that*

$$(28) \quad \left\{ \varrho e^{it} : t \in \left[0, \frac{2\pi}{l-k}\right), \varrho \in \sqrt{Z_r} \right\} \subset Z.$$

Then for any system $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ the following conditions are equivalent:

- (a) $\{c_{m,n}\}_{(m,n) \in T}$ satisfies (iv)_Z,
- (b) the sequence $\{c_{m,m}\}_{m=0}^\infty$ satisfies the *Riesz-Haviland positivity condition* on Z_r , $c_{l,k} = \overline{c_{k,l}}$ and $|c_{k,l}| \leq c_{k+\varkappa, k+\varkappa}$.

In particular, if $\mathcal{Z}\left(\frac{2\pi}{l-k}\right) \subset Z \subsetneq \mathbb{C}$, then the answer to the question 2° is in the affirmative.

PROOF. (a) \Rightarrow (b) The Riesz-Haviland positivity condition for $\{c_{m,m}\}_{m=0}^\infty$ has been already discussed in the proof of Lemma 25. Notice that for every $\theta \in \mathbb{C}$ such that $|\theta| \leq 1$, the polynomial

$$2(z\bar{z})^{k+\varkappa} + \theta z^l \bar{z}^k + \bar{\theta} z^k \bar{z}^l = 2(z\bar{z})^{k+\varkappa} + 2\Re(\theta z^l \bar{z}^k)$$

is nonnegative on \mathbb{C} . Hence, by (a), $2c_{k+\varkappa, k+\varkappa} + \theta c_{l,k} + \bar{\theta} c_{k,l} \geq 0$ for all $\theta \in \mathbb{C}$ with $|\theta| \leq 1$. Substituting successively $\theta = 0$, $\theta = 1$ and $\theta = i$, we deduce that

$c_{k+\varkappa, k+\varkappa} \geq 0$ and $c_{l,k} = \overline{c_{k,l}}$. In turn, taking θ such that $|\theta| = 1$ and $\theta c_{l,k} = -|c_{l,k}|$, we obtain the remaining inequality in (b).

(b) \Rightarrow (a) Assume that $p(z, \bar{z}) = \sum_{j=0}^N p_j z^j \bar{z}^j + \theta z^l \bar{z}^k + \tilde{\theta} z^k \bar{z}^l \geq 0$ for all $z \in Z$ ($p_0, \dots, p_N, \theta, \tilde{\theta} \in \mathbb{C}$). Since Z is a determining set for $\mathbb{C}[z, \bar{z}]$, we see that $\tilde{\theta} = \bar{\theta}$ and $p_j \in \mathbb{R}$ for all j . As $\varrho e^{it} \in Z$ for all $t \in [0, \frac{\pi}{\varkappa})$ and $\varrho \in \sqrt{Z_r}$, we get

$$\sum_{j=0}^N p_j \varrho^{2j} + 2\varrho^{2(k+\varkappa)} \Re(\theta e^{2i\varkappa t}) = p(\varrho e^{it}, \varrho e^{-it}) \geq 0, \quad \varrho \in \sqrt{Z_r}, t \in \left[0, \frac{\pi}{\varkappa}\right).$$

Since the numbers $2\varkappa t$, $t \in [0, \frac{\pi}{\varkappa})$, exhaust the whole interval $[0, 2\pi)$, we deduce that $\sum_{j=0}^N p_j \varrho^{2j} - 2|\theta| \varrho^{2(k+\varkappa)} \geq 0$ for all $\varrho \in \sqrt{Z_r}$. By the Riesz-Haviland positivity condition, we see that $\sum_{j=0}^N p_j c_{j,j} - 2|\theta| c_{k+\varkappa, k+\varkappa} \geq 0$. Owing to this inequality and (b), we conclude that

$$-\theta c_{l,k} - \tilde{\theta} c_{k,l} = -2 \Re(\theta c_{l,k}) \leq 2|\theta| |c_{l,k}| \leq 2|\theta| c_{k+\varkappa, k+\varkappa} \leq \sum_{j=0}^N p_j c_{j,j},$$

which shows that $\{c_{m,n}\}_{(m,n) \in T}$ satisfies (iv) $_Z$.

The “in particular” part of the conclusion follows from Lemma 17 and the equivalence (a) \Leftrightarrow (b) (because $Z_r = \mathbb{C}_r = [0, \infty)$). \square

Note that if $l-k$ is even, then by Theorem A and the measure transport theorem the condition (b) of Proposition 26 is equivalent to the condition (b) below. The key observation is that the integral $\int_{[0, \infty)} \varrho^{k+l} d\nu(\varrho)$ is equal to $c_{k+\varkappa, k+\varkappa}$.

PROPOSITION 27. *Let $T = \mathcal{T}_{k,l}$ with $l > k$ and let Z be a closed subset of \mathbb{C} satisfying (28). Given a system $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$, consider the following two conditions:*

- (a) $\{c_{m,n}\}_{(m,n) \in T}$ satisfies (iv) $_Z$,
- (b) *there exists a finite positive Borel measure ν on $[0, \infty)$ supported in $\sqrt{Z_r}$ such that $c_{m,m} = \int_{[0, \infty)} \varrho^{2m} d\nu(\varrho)$ for all $m \in \mathbb{Z}_+$, $c_{l,k} = \overline{c_{k,l}}$ and $|c_{k,l}| \leq \int_{[0, \infty)} \varrho^{k+l} d\nu(\varrho)$.*

Then (b) implies (a). If additionally $Z_r = [0, \infty)$ or $\{c_{m,m}\}_{m=0}^\infty$ is a determinate Stieltjes moment sequence, then (a) implies (b). In particular, if $\mathcal{Z}\left(\frac{2\pi}{l-k}\right) \subset Z \subsetneq \mathbb{C}$ with $l-k \geq 2$, then the answer to the question 2 $^\circ$ is in the affirmative.

PROOF. (a) \Rightarrow (b) Since $\{c_{m,n}\}_{(m,n) \in T}$ evidently satisfies (iv), we deduce from Theorem 20 that there exists a positive Borel measure μ on \mathbb{C} such that $c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n d\mu(z)$ for all $(m,n) \in T$. Clearly, $c_{l,k} = \overline{c_{k,l}}$. Applying the measure transport theorem, we see that the finite positive Borel measure ν on $[0, \infty)$ defined via $\nu(\sigma) = \mu(\{z \in \mathbb{C} : |z| \in \sigma\})$ for Borel subsets σ of $[0, \infty)$ satisfies the first equality in (b). The inequality in (b) can be justified as follows:

$$|c_{k,l}| \leq \int_{\mathbb{C}} |z|^{k+l} d\mu(z) = \int_{[0, \infty)} \varrho^{k+l} d\nu(\varrho).$$

Thus the case of $Z_r = [0, \infty)$ is settled. If $Z_r \subsetneq [0, \infty)$ and $\{c_{m,m}\}_{m=0}^\infty$ is a determinate Stieltjes moment sequence, then by Lemma 25 and the measure transport theorem we deduce that the measure ν is supported in $\sqrt{Z_r}$.

(b) \Rightarrow (a) By the inequality in (b), there exists $\theta \in \mathbb{C}$ such that $|\theta| \leq 1$ and $c_{k,l} = \theta \int_{[0,\infty)} \varrho^{k+l} d\nu(\varrho)$. It is easily seen that there exist (not necessarily distinct) numbers $t_1, t_2 \in [0, 2\pi)$ such that $\bar{\theta} = \frac{1}{2}(e^{it_1} + e^{it_2})$. Let ζ be a positive Borel measure on $[0, 2\pi)$ supported in $\{\frac{t_1}{j}, \frac{t_2}{j}\}$ with $\zeta(\{\frac{t_1}{j}\}) = \zeta(\{\frac{t_2}{j}\}) = \frac{1}{2}$, where $j = l - k$. Define the Borel measure μ on \mathbb{C} via

$$\mu(\sigma) = \int_{[0,2\pi)} \int_{[0,\infty)} \chi_\sigma(\varrho e^{it}) d\nu(\varrho) d\zeta(t), \quad \sigma - \text{Borel subset of } \mathbb{C}.$$

It is a matter of routine to verify that $c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n d\mu(z)$ for all $(m, n) \in T$ (hint: $\bar{\theta} = \int_{[0,2\pi)} e^{ijt} d\zeta(t)$). One can show that the closed support of the measure μ is contained in the set

$$(29) \quad \{\varrho e^{it_1/j} : \varrho \in \sqrt{Z_r}\} \cup \{\varrho e^{it_2/j} : \varrho \in \sqrt{Z_r}\},$$

which in view of (28) is a subset of Z . This implies that the system $\{c_{m,n}\}_{(m,n) \in T}$ satisfies (iv) $_Z$. Observe that the construction of the measure μ is based on the possibility of representing θ as an arithmetic mean of two complex numbers of absolute value 1. In fact, the same proof works if θ is represented as a finite convex combination of complex numbers of absolute value 1, in which case the closed support of μ consists of a finite number of sets of the type appearing in (29).

The “in particular” part of the conclusion follows from the equivalence (a) \Leftrightarrow (b) which is valid because $Z_r = \mathbb{C}_r = [0, \infty)$. \square

The following proposition shows that the angle $\frac{2\pi}{l-k}$ appearing in the assumption (28) of Propositions 26 and 27 is optimal, i.e. it cannot be made smaller. It is worth pointing out that if $l = k + 1$, then the assumption (30) below is satisfied by any proper subset Z of \mathbb{C} .

PROPOSITION 28. *Let T be a symmetric subset of \mathfrak{N} such that $\mathcal{T}_{k,l} \subset T$ for some integers $l > k \geq 0$ and let Z be a closed subset of \mathbb{C} for which there exists $\lambda \in \mathbb{C}$ such that*

$$(30) \quad \{z \in \mathbb{C} : z^{l-k} = \lambda\} \subset \mathbb{C} \setminus Z.$$

Then there exists a system $\{c_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ which satisfies (iv), but not (iv) $_Z$. In particular, if $Z \subset \mathcal{Z}(\alpha)$ with $\alpha \in [0, \frac{2\pi}{l-k})$, then the answer to the question 2 $^\circ$ is in the negative.

PROOF. Put $j = l - k$. Without loss of generality we may assume that $\lambda \neq 0$. Then there exists $\varepsilon > 0$ such that

$$(31) \quad \emptyset \neq \Delta_\varepsilon \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z^j - \lambda| < \varepsilon\} \subset \mathbb{C} \setminus (Z \cup \{0\}).$$

To see this suppose that $\lambda_1, \dots, \lambda_j$ are all the complex j -roots of λ . By (30) and the closedness of Z , there exists $\delta > 0$ such that $\min_{n \in \{1, \dots, j\}} |z - \lambda_n| \geq \delta$ for all $z \in Z \cup \{0\}$. Since $z^j - \lambda = \prod_{n=1}^j (z - \lambda_n)$ and consequently

$$\left(\min_{n \in \{1, \dots, j\}} |z - \lambda_n| \right)^j \leq \prod_{n=1}^j |z - \lambda_n| = |z^j - \lambda|, \quad z \in \mathbb{C},$$

we deduce that Δ_ε is contained in $\mathbb{C} \setminus (Z \cup \{0\})$ whenever $\varepsilon \leq \delta^j$.

Consider a nonzero finite positive Borel measure μ on \mathbb{C} compactly supported in the open set Δ_ε . Set $c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n d\mu(z)$ for $(m, n) \in T$ and

$$p(z, \bar{z}) = |z|^{2k}(|z^j - \lambda|^2 - \varepsilon^2) = z^l \bar{z}^l - \bar{\lambda} z^l \bar{z}^k - \lambda z^k \bar{z}^l + (|\lambda|^2 - \varepsilon^2) z^k \bar{z}^k, \quad z \in \mathbb{C}.$$

Plainly, $p \in \mathbb{C}_T[z, \bar{z}]$. By (31), we see that $p(z, \bar{z}) \geq 0$ for all $z \in Z$ and $p(z, \bar{z}) < 0$ for all $z \in \Delta_\varepsilon$ (because $0 \notin \Delta_\varepsilon$). Evidently, $\{c_{m,n}\}_{(m,n) \in T}$ satisfies (iv), but not (iv) $_Z$, because $\mu \neq 0$ and

$$c_{l,l} - \bar{\lambda} c_{l,k} - \lambda c_{k,l} + (|\lambda|^2 - \varepsilon^2) c_{k,k} = \int_{\Delta_\varepsilon} p(z, \bar{z}) d\mu(z) < 0.$$

To prove the “in particular” part of the conclusion note that $\lambda = e^{ij\theta}$ satisfies (30) for any $\theta \in (\alpha, 2\pi/j)$. The proof is complete. \square

Summing up, the case of sets $\mathcal{T}_{k,l}$ serves as a good elucidation of the interplay between T and Z which is crucial when dealing with the question 2°. In the table below we gather information concerning this question; we keep the assumptions on T and Z made therein.

Answer	T	Z
NO	arbitrary	$Z_r \subsetneq [0, \infty)$
	$T \supset \mathcal{T}_{k,l}, l - k \geq 1$	Z satisfies (30)
	$T \supset \mathcal{T}_{k,l}, l - k \geq 1$	$Z \subset \mathcal{Z}(\alpha), \alpha \in [0, \frac{2\pi}{l-k})$
	$T \supset \mathcal{T}_{k,k+1}, k \geq 0$	arbitrary
YES	$T = \mathcal{T}_{k,l}, l - k \geq 2$	$Z \supset \mathcal{Z}(\frac{2\pi}{l-k})$
	$T = (\mathfrak{N}_+)_\mathfrak{h}$	$Z_r = [0, \infty)$

To justify the ‘YES’ part of the table one should notice that if $T' \subset T$, $Z \subset Z'$ and the answer to the question 2° is in the affirmative for T and Z , then it is so for T' and Z' . In turn, the ‘NO’ part requires contraposition, i.e. if the answer to 2° is in the negative for T' and Z' , then it is so for T and Z . In view of these properties and the table above, if $T = \mathcal{T}_{k,l}$ with $l - k \geq 1$ and $Z = \mathcal{Z}(\alpha)$, then the answer to the question 2° is in the negative for $\alpha \in [0, \frac{2\pi}{l-k})$ and in the affirmative for $\alpha \in [\frac{2\pi}{l-k}, 2\pi)$.

6. Subnormality. Let S be a densely defined linear operator in a complex Hilbert space \mathcal{H} with domain $\mathcal{D}(S)$. We say that S is *subnormal* if there exist a complex Hilbert space \mathcal{K} and a normal operator N in \mathcal{K} such that $\mathcal{H} \subset \mathcal{K}$ (isometric embedding), $\mathcal{D}(S) \subset \mathcal{D}(N)$ and $Sh = Nh$ for all $h \in \mathcal{D}(S)$. For fundamentals of the theory of unbounded subnormal operators we refer the reader to [52, 53, 54].

The following characterization of subnormality simplifies substantially that of [53, Theorem 3] (one double sum turns out to be redundant). As in [53], it is intrinsic in a sense that no extension is involved. Theorem 29 is also related to part (iv) of [55, Theorem 37] which in the case of $\mathcal{F} = \mathcal{D}$ is equivalent to condition (iii) below.

THEOREM 29. *Let S be a densely defined linear operator in a complex Hilbert space \mathcal{H} such that $S(\mathcal{D}(S)) \subset \mathcal{D}(S)$. If $Z \subset \mathbb{C}_*$ is a determining set for $\mathbb{C}[z, \bar{z}]$, then the following conditions are equivalent:*

- (i) S is subnormal,

(ii) for every system $\{a_{p,q}^{i,j}\}_{p,q=0,\dots,n}^{i,j=1,\dots,m} \subset \mathbb{C}$, if

$$(32) \quad \sum_{i,j=1}^m \sum_{p,q=0}^n a_{p,q}^{i,j} \lambda^p \bar{\lambda}^q z_i \bar{z}_j \geq 0, \quad \lambda, z_1, \dots, z_m \in \mathbb{C},$$

then

$$(33) \quad \sum_{i,j=1}^m \sum_{p,q=0}^n a_{p,q}^{i,j} \langle S^p f_i, S^q f_j \rangle \geq 0, \quad f_1, \dots, f_m \in \mathcal{D}(S),$$

(iii) for every system $\{a_{p,q}^{i,j}\}_{p,q=0,\dots,n}^{i,j=1,\dots,m} \subset \mathbb{C}$, if there is a finite matrix $[q_{i,l}]_{i=1}^m \stackrel{k}{l=1}$ with entries in $\mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$ such that

$$(34) \quad \sum_{p,q=0}^n a_{p,q}^{i,j} \lambda^p \bar{\lambda}^q = \sum_{l=1}^k q_{i,l}(\lambda, \bar{\lambda}) \overline{q_{j,l}(\lambda, \bar{\lambda})}, \quad \lambda \in Z, i, j = 1, \dots, m,$$

then (33) holds.

PROOF. The proof of implication (i) \Rightarrow (ii) proceeds along the same lines as the proof of the “only if” part of [53, Theorem 3]. The other possibility is to argue as in the proof of Lemma 12 (i).

(ii) \Rightarrow (iii) First note that if (34) holds, then, by the determining property of Z (cf. Lemma 19), the equality in (34) is valid for all $\lambda \in \mathbb{C}_*$. It is then easily seen that (32) holds, which by (ii) yields (33).

(iii) \Rightarrow (i) Consider the $*$ -semigroup $\mathfrak{S} = \mathfrak{N}_+$ (which is operator semiperfect due to Remark 7 and Proposition 23 in [55]), the linear space $\mathcal{D} = \mathcal{D}(S)$ and the sets $T = \mathbb{Z}_+ \times \mathbb{Z}_+$ and $Y = Y_Z$ (cf. (19)). Define the mapping $\Phi: T \rightarrow \mathcal{S}(\mathcal{D})$ by

$$\Phi(m, n)(f, g) = \langle S^m f, S^n g \rangle, \quad f, g \in \mathcal{D}, (m, n) \in T,$$

and attach to it the linear mapping $\Lambda_{\Phi, Y}: \mathcal{P}_T(Y) \rightarrow \mathcal{S}(\mathcal{D})$ via formula (12). Then a simple calculation based on Lemma 6 shows that (iii) is equivalent to the complete f -positivity of $\Lambda_{\Phi, Y}$. Applying implication (iv) \Rightarrow (i) of Theorem 14 and implication (iii) \Rightarrow (i) of [55, Theorem 37] with $\mathcal{F} = \mathcal{D}$ completes the proof. \square

7. Unitary dilation of several contractions. In what follows \varkappa stands for a positive integer. Set $\mathbb{Z}_- = -\mathbb{Z}_+$. Denote by \mathbb{Z}^\varkappa , \mathbb{Z}_+^\varkappa , \mathbb{Z}_-^\varkappa and \mathbb{T}^\varkappa the cartesian product of \varkappa copies of \mathbb{Z} , \mathbb{Z}_+ , \mathbb{Z}_- and \mathbb{T} , respectively. For simplicity, we write 0 for the zero element of the group \mathbb{Z}^\varkappa . Observe that $\mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa = \mathbb{Z}^\varkappa$ only for $\varkappa = 1$. Let $\mathcal{P}(\mathbb{T}^\varkappa)$ stand for the linear space of all functions $p: \mathbb{T}^\varkappa \rightarrow \mathbb{C}$ of the form

$$(35) \quad p(z) = \sum_{\alpha \in \mathbb{Z}^\varkappa} a_\alpha z^\alpha, \quad z \in \mathbb{T}^\varkappa,$$

where $\{a_\alpha\}_{\alpha \in \mathbb{Z}^\varkappa}$ is a finite system of complex numbers, and $z^\alpha = z_1^{\alpha_1} \dots z_\varkappa^{\alpha_\varkappa}$ for $z = (z_1, \dots, z_\varkappa) \in \mathbb{T}^\varkappa$ and $\alpha = (\alpha_1, \dots, \alpha_\varkappa) \in \mathbb{Z}^\varkappa$. The members of $\mathcal{P}(\mathbb{T}^\varkappa)$ are called *trigonometric polynomials* in \varkappa variables. A trigonometric polynomial p vanishes on \mathbb{T}^\varkappa if and only if all its coefficients a_α vanish. Given $T \subset \mathbb{Z}^\varkappa$, we denote by $\mathcal{P}_T(\mathbb{T}^\varkappa)$ the linear space of all trigonometric polynomials $p \in \mathcal{P}(\mathbb{T}^\varkappa)$ of the form (35), where $a_\alpha = 0$ for all $\alpha \in \mathbb{Z}^\varkappa \setminus T$. We abbreviate $\mathcal{P}_{\mathbb{Z}_+^\varkappa}(\mathbb{T}^\varkappa)$ to $\mathcal{P}_+(\mathbb{T}^\varkappa)$ and $\mathcal{P}_{\mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa}(\mathbb{T}^\varkappa)$ to $\mathcal{P}_\pm(\mathbb{T}^\varkappa)$. One can think of members of $\mathcal{P}_+(\mathbb{T}^\varkappa)$ as *analytic* trigonometric polynomials. A nonempty subset Y of \mathbb{T}^\varkappa is said to be *determining* for $\mathcal{P}(\mathbb{T}^\varkappa)$ (respectively: $\mathcal{P}_+(\mathbb{T}^\varkappa)$) if each trigonometric polynomial

$p \in \mathcal{P}(\mathbb{T}^\varkappa)$ (respectively: $p \in \mathcal{P}_+(\mathbb{T}^\varkappa)$) vanishing on Y vanishes on the whole set \mathbb{T}^\varkappa . Note that any infinite subset of \mathbb{T} is determining for $\mathcal{P}(\mathbb{T})$.

LEMMA 30. *A subset Y of \mathbb{T}^\varkappa is determining for $\mathcal{P}(\mathbb{T}^\varkappa)$ if and only if it is determining for $\mathcal{P}_+(\mathbb{T}^\varkappa)$.*

PROOF. This is clear, because for every $p \in \mathcal{P}(\mathbb{T}^\varkappa)$, there exists $n \in \mathbb{Z}_+$ such that the trigonometric polynomial $z_1^n \dots z_\varkappa^n p(z_1, \dots, z_\varkappa)$ is analytic. \square

Let $\mathbf{A} = (A_1, \dots, A_\varkappa)$ be a \varkappa -tuple of bounded linear operators on a complex Hilbert space \mathcal{H} . Define the family $\{\mathbf{A}^{[\alpha]} : \alpha \in \mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa\}$ by

$$\mathbf{A}^{[\alpha]} = \begin{cases} A_1^{\alpha_1} \dots A_\varkappa^{\alpha_\varkappa}, & \alpha \in \mathbb{Z}_+^\varkappa, \\ A_1^{*|\alpha_1|} \dots A_\varkappa^{*|\alpha_\varkappa|}, & \alpha \in \mathbb{Z}_-^\varkappa. \end{cases}$$

For $\alpha \in \mathbb{Z}_+^\varkappa$ we replace $\mathbf{A}^{[\alpha]}$ by the standard multi-index notation \mathbf{A}^α . Following [64, page 32], we say that a \varkappa -tuple \mathbf{A} has a *unitary power dilation* if there exists a complex Hilbert space $\mathcal{K} \supset \mathcal{H}$ (isometric embedding) and a \varkappa -tuple $\mathbf{U} = (U_1, \dots, U_\varkappa)$ of commuting unitary operators on \mathcal{K} such that

$$\mathbf{A}^\alpha = P\mathbf{U}^\alpha|_{\mathcal{H}}, \quad \alpha \in \mathbb{Z}_+^\varkappa,$$

where P stands for the orthogonal projection of \mathcal{K} onto \mathcal{H} . Such \mathbf{U} is called a *unitary power dilation* of \mathbf{A} . The proof of the following fact is left to the reader.

LEMMA 31. *If \mathbf{U} is a unitary power dilation of \mathbf{A} , then the operators A_1, \dots, A_\varkappa commute if and only if $\mathbf{A}^{[\alpha]} = P\mathbf{U}^\alpha|_{\mathcal{H}}$ for all $\alpha \in \mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa$.*

We are now in a position to formulate necessary and sufficient conditions for a \varkappa -tuple of bounded operators to have a unitary power dilation. Theorem 32 below is related to characterizations of families of operators having unitary power dilations given in [56, Corollary 6] (see also [35, Lemma 1] for another formulation which does not appeal to the boundedness of \mathbf{A} ; in fact, one can easily write a version of Theorem 32 for operators which are not a priori assumed to be bounded). Different approaches to the problem of the existence of unitary power dilation have recently appeared in [65] and [2]; however, the solutions proposed therein are not written in terms of operators in question.

THEOREM 32. *If $\mathbf{A} = (A_1, \dots, A_\varkappa)$ is a \varkappa -tuple of bounded linear operators on a complex Hilbert space \mathcal{H} and Y is a determining set for $\mathcal{P}_+(\mathbb{T}^\varkappa)$, then the following conditions are equivalent:*

- (i) \mathbf{A} has a unitary power dilation and the operators A_1, \dots, A_\varkappa commute,
- (ii) for every finite system $\{a_\alpha^{i,j} : i, j = 1, \dots, m, \alpha \in \mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa\} \subset \mathbb{C}$, if

$$(36) \quad \sum_{i,j=1}^m \sum_{\alpha \in \mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa} a_\alpha^{i,j} \lambda^\alpha z_i \bar{z}_j \geq 0, \quad \lambda \in \mathbb{T}^\varkappa, z_1, \dots, z_m \in \mathbb{C},$$

then

$$(37) \quad \sum_{i,j=1}^m \sum_{\alpha \in \mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa} a_\alpha^{i,j} \langle \mathbf{A}^{[\alpha]} f_i, f_j \rangle \geq 0, \quad f_1, \dots, f_m \in \mathcal{H},$$

- (iii) for every finite system $\{a_\alpha^{i,j} : i, j = 1, \dots, m, \alpha \in \mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa\} \subset \mathbb{C}$, if there is a finite matrix $[q_{i,l}]_{i=1}^m [l=1]^k$ with entries in $\mathcal{P}_+(\mathbb{T}^\varkappa)$ such that

$$(38) \quad \sum_{\alpha \in \mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa} a_\alpha^{i,j} \lambda^\alpha = \sum_{l=1}^k q_{i,l}(\lambda) \overline{q_{j,l}(\lambda)}, \quad \lambda \in Y, i, j = 1, \dots, m,$$

then (37) holds.

PROOF. Without loss of generality we can assume that Y is a determining set for $\mathcal{P}(\mathbb{T}^\varkappa)$ (cf. Lemma 30). In what follows, we regard \mathbb{Z}^\varkappa as the $*$ -semigroup equipped with coordinatewise defined addition as semigroup operation and involution $\alpha^* = -\alpha$, $\alpha \in \mathbb{Z}^\varkappa$. In turn, \mathbb{T}^\varkappa is regarded as the multiplicative $*$ -semigroup with coordinatewise defined multiplication as semigroup operation and involution

$$(z_1, \dots, z_\varkappa)^* = (\bar{z}_1, \dots, \bar{z}_\varkappa), \quad (z_1, \dots, z_\varkappa) \in \mathbb{T}^\varkappa.$$

It is easily checked that the dual $*$ -semigroup $\mathfrak{X}_{\mathbb{Z}^\varkappa}$ of \mathbb{Z}^\varkappa can be identified algebraically with the $*$ -semigroup \mathbb{T}^\varkappa via the mapping

$$(39) \quad \mathfrak{X}_{\mathbb{Z}^\varkappa} \ni \chi \mapsto (\chi(e_1), \dots, \chi(e_\varkappa)) \in \mathbb{T}^\varkappa,$$

where $e_j = (\delta_{j,1}, \dots, \delta_{j,\varkappa}) \in \mathbb{Z}^\varkappa$ ($\delta_{i,j}$ is the Kronecker delta function). Under this identification, $\hat{\alpha}$ is given by

$$\hat{\alpha}(z) = z^\alpha, \quad z = (z_1, \dots, z_\varkappa) \in \mathbb{T}^\varkappa, \alpha \in \mathbb{Z}^\varkappa,$$

which means that the notation $\mathcal{P}(\mathbb{T}^\varkappa)$ introduced at the beginning of this section is consistent with that for $*$ -semigroups in Section 1, and that Y is a determining subset of $\mathfrak{X}_{\mathbb{Z}^\varkappa}$. It is well known that the $*$ -semigroup \mathbb{Z}^\varkappa is operator semiperfect (e.g. see [38] and footnote 7). Put $T = \mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa$ and define $\Phi: T \rightarrow \mathcal{S}(\mathcal{H})$ via

$$\Phi(\alpha)(f, g) = \langle \mathbf{A}^{[\alpha]} f, g \rangle, \quad f, g \in \mathcal{H}, \alpha \in T.$$

In view of Lemma 31 and [56, Theorem 3], the current condition (i) is equivalent to condition (i) of Theorem 14 with $\mathfrak{S} = \mathbb{Z}^\varkappa$ and $\mathcal{D} = \mathcal{H}$. In turn, the current condition (ii) is a counterpart of condition (iii) of Theorem 14. Finally, by Lemma 6, the current condition (iii) is a counterpart of condition (iv) of Theorem 14, because if (38) holds for some matrix $[q_{i,l}]_{i=1}^m [l=1]^k$ with entries in $\mathcal{P}(\mathbb{T}^\varkappa)$, then there exists $n \in \mathbb{Z}_+$ such that all the trigonometric polynomials

$$(40) \quad \tilde{q}_{i,l}(z_1, \dots, z_\varkappa) \stackrel{\text{def}}{=} z_1^n \dots z_\varkappa^n q_{i,l}(z_1, \dots, z_\varkappa), \quad z_1, \dots, z_\varkappa \in \mathbb{T},$$

are analytic and (38) is valid with $[\tilde{q}_{i,l}]_{i=1}^m [l=1]^k$ in place of $[q_{i,l}]_{i=1}^m [l=1]^k$. Hence, applying Theorem 14 completes the proof. \square

REMARK 33. The implication (iii) \Rightarrow (ii) of Theorem 32 can also be deduced from [22, Corollary 5.2]. Indeed, if (36) is valid, then for every real $\varepsilon > 0$, the square-matrix-valued trigonometric polynomial $Q^{(\varepsilon)}(\lambda) \stackrel{\text{def}}{=} \sum_{\alpha \in \mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa} Q_\alpha \lambda^\alpha + \varepsilon I_m$, where $Q_\alpha = [a_\alpha^{i,j}]_{i,j=1}^m$ and $I_m = [\delta_{i,j}]_{i,j=1}^m$, is strictly positive on \mathbb{T}^\varkappa . By [22, Corollary 5.2], the polynomial $Q^{(\varepsilon)}$ has a factorization by an analytic (in general non-square) matrix-valued trigonometric polynomial. This implies that the polynomial $Q^{(\varepsilon)}$ takes the form which is required in (38) (use the trick⁹ from the proof of Lemma

⁹ This additional effort comes from the fact that Dritschel's factorization $F(\lambda)^* F(\lambda)$ differs from the factorization $P(\lambda) P(\lambda)^*$ required in (38) by the location of the asterisk.

30). Hence, by (iii), we have

$$\sum_{i,j=1}^m \sum_{\alpha \in \mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa} a_\alpha^{i,j} \langle \mathbf{A}^{[\alpha]} f_i, f_j \rangle + \varepsilon \sum_{i=1}^m \|f_i\|^2 \geq 0, \quad f_1, \dots, f_m \in \mathcal{H}, \varepsilon > 0.$$

Passing with ε to 0 completes the proof.

8. The truncated multidimensional trigonometric moment problem.

The problem in the title goes back to Krein's theorem [36] on extending positive definite functions from an interval to \mathbb{R} . A several dimensional version of this is not true according to examples of Calderón and Pepinsky [17] (the additive group \mathbb{Z}^\varkappa) and Rudin [45] (the additive groups \mathbb{Z}^\varkappa and \mathbb{R}^\varkappa). In this section we concentrate on the discrete case. Our result, which is related to [56, Corollary 4], deals with the truncated multidimensional trigonometric moment problem on symmetric subsets of \mathbb{Z}^\varkappa .

It was proved in [17], and independently in [45], that a finite nonempty subset Λ of \mathbb{Z}^\varkappa has the extension property (i.e. each complex function on $\Lambda - \Lambda$ which is positive definite on Λ extends to a positive definite complex function on \mathbb{Z}^\varkappa) if and only if each nonnegative trigonometric polynomial in $\mathcal{P}_{\Lambda - \Lambda}(\mathbb{T}^\varkappa)$ is equal to a finite sum of squares of moduli of trigonometric polynomials in $\mathcal{P}_\Lambda(\mathbb{T}^\varkappa)$. Recently, Gabardo found new conditions under which Λ has or fails to have the extension property (cf. [27]; see also [28] for Λ -determinacy and [25, 26] for related questions). Clearly, if Λ has the extension property, then $\mathcal{P}_{\Lambda - \Lambda}^+(\mathbb{T}^\varkappa) = \Sigma_{\Lambda - \Lambda}^2(\mathbb{T}^\varkappa)$ (see Section 9 for notation). Note that each *difference set* $T \subset \mathbb{Z}^\varkappa$, i.e. a set of the form $\Lambda - \Lambda$ with some nonempty $\Lambda \subset \mathbb{R}^\varkappa$, has the property $0 \in T = -T$, which is required in Theorem 34 below. However, not every set $T \subset \mathbb{Z}^\varkappa$ satisfying $0 \in T = -T$ is a difference set, which can be seen even for $\varkappa = 1$. Namely, one can show that for all integers k, n such that $1 \leq k < n - k < n$ (necessarily $n \geq 3$ and $k < n/2$) any subset T of \mathbb{Z} fulfilling the following conditions

- (i) $0 \in T = -T$,
- (ii) $T \subset \{j \in \mathbb{Z} : |j| \leq n\}$,
- (iii) $n - k, n \in T$,
- (iv) $k \notin T$ and $j \notin T$ for every integer j such that $n - k < j < n$,

is not a difference set (hint: replace Λ by $\Lambda - \min \Lambda$). The cardinality of such sets T may (and does) vary between 5 and $2(n - k) + 1$. There is a simple way of producing multidimensional variants of non-difference sets from one-dimensional ones. Indeed, if T_1 is a non-difference subset of \mathbb{Z} and T' is any subset of \mathbb{Z}^\varkappa , then $T_1 \times T'$ is a non-difference subset of $\mathbb{Z}^{\varkappa+1}$ (hint: $P(\Lambda - \Lambda) = P(\Lambda) - P(\Lambda)$, where $P(n, \alpha) = n$ for $n \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}^\varkappa$). Of course, there are non-difference sets which cannot be obtained this way, e.g. $\varkappa = 2$ and $T = \{(0, 0), (1, 0), (-1, 0), (1, 1), (-1, -1)\}$. Summing up, our solutions of the truncated trigonometric moment problem given in Theorem 34 below allow for much more general truncations, even in the case of finite data T . Surprisingly, the infinite set $\mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa$, playing a pivotal role in Section 7, is a difference set. Indeed, if $\mathbb{Z}_+ \ni n \mapsto \alpha_n \in \mathbb{Z}_+^\varkappa$ is any surjection with $\alpha_0 = 0$, then $\mathbb{Z}_+^\varkappa \cup \mathbb{Z}_-^\varkappa = \Lambda - \Lambda$ with

$$\Lambda = \{\alpha_0 + \dots + \alpha_n : n \in \mathbb{Z}_+\}.$$

Another (more explicit) choice of Λ for $\varkappa = 2$ is illustrated in Figure 2.

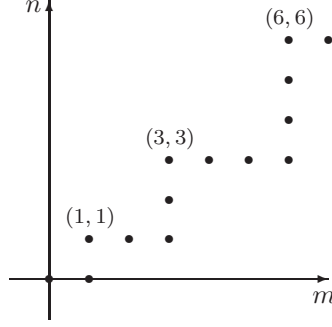


Figure 2. An example of Λ such that $\mathbb{Z}_+^2 \cup \mathbb{Z}_-^2 = \Lambda - \Lambda$.

We now go back to solving the truncated trigonometric moment problem.

THEOREM 34. *Assume that T is a subset of \mathbb{Z}^\varkappa such that¹⁰ $0 \in T = -T$, and Y is a determining set for $\mathcal{P}_+(\mathbb{T}^\varkappa)$. If $\{c_\alpha\}_{\alpha \in T}$ is a sequence of complex numbers, then the following conditions are equivalent:*

- (i) *there exists a finite positive Borel measure μ on \mathbb{T}^\varkappa such that*

$$(41) \quad c_\alpha = \int_{\mathbb{T}^\varkappa} z^\alpha d\mu(z), \quad \alpha \in T,$$

- (ii) *$\sum_{\alpha \in T} a_\alpha c_\alpha \geq 0$ for every finite system $\{a_\alpha\}_{\alpha \in T}$ of complex numbers such that $\sum_{\alpha \in T} a_\alpha z^\alpha \geq 0$ for all $z \in \mathbb{T}^\varkappa$,*
 (iii) *$\sum_{\alpha \in T} a_\alpha c_\alpha \geq 0$ for every finite system $\{a_\alpha\}_{\alpha \in T}$ of complex numbers for which there exist finitely many analytic trigonometric polynomials $q_1, \dots, q_k \in \mathcal{P}_+(\mathbb{T}^\varkappa)$ such that $\sum_{\alpha \in T} a_\alpha z^\alpha = \sum_{j=1}^k |q_j(z)|^2$ for all $z \in Y$.*

PROOF. We can argue essentially as in the proof of Theorem 32 using Theorem 15 instead of Theorem 14. \square

It is worth mentioning that Proposition 24 can be easily adapted to the present context. We say that a locally convex topology τ on the linear space $\mathcal{P}_\pm(\mathbb{T}^\varkappa)$ (cf. Section 7) is *evaluable* if the set of all points $\lambda \in \mathbb{T}^\varkappa$ for which the evaluation

$$\mathcal{P}_\pm(\mathbb{T}^\varkappa) \ni p \mapsto p(\lambda) \in \mathbb{C}$$

is τ -continuous is dense in \mathbb{T}^\varkappa . Given $Y \subset \mathbb{T}^\varkappa$, we denote by $\Sigma^2(Y)$ the set of all trigonometric polynomials $q \in \mathcal{P}_\pm(\mathbb{T}^\varkappa)$ for which there exist finitely many analytic trigonometric polynomials $q_1, \dots, q_n \in \mathcal{P}_+(\mathbb{T}^\varkappa)$ such that

$$q(z) = \sum_{j=1}^n |q_j(z)|^2, \quad z \in Y.$$

Arguing as in the proof of Proposition 24 with $p_\varepsilon \in \mathcal{P}_\pm(\mathbb{T}^\varkappa)$ given by

$$p_\varepsilon(z) = \sum_{j=1}^\varkappa |z_j - \lambda_j|^2 - \varepsilon^2 = 2\varkappa - \varepsilon^2 - \sum_{j=1}^\varkappa \bar{\lambda}_j z_j - \sum_{j=1}^\varkappa \lambda_j z_j^{-1}, \quad z = (z_1, \dots, z_\varkappa) \in \mathbb{T}^\varkappa,$$

where $(\lambda_1, \dots, \lambda_\varkappa) \in \mathbb{T}^\varkappa \setminus Y$, we are led to the ensuing result.

¹⁰ This is the explicit form of condition (14) under the circumstances of the $*$ -semigroup \mathbb{Z}^\varkappa .

PROPOSITION 35. *Let Y be a closed proper subset of \mathbb{T}^∞ which is determining for $\mathcal{P}_+(\mathbb{T}^\infty)$ and let τ be an evaluable locally convex topology on $\mathcal{P}_\pm(\mathbb{T}^\infty)$. Then there exists a trigonometric polynomial $p \in \mathcal{P}_\pm(\mathbb{T}^\infty)$ which is nonnegative on Y and which does not belong to the τ -closure of $\Sigma^2(Y)$. In particular, p is not in $\Sigma^2(Y)$.*

Note that if a trigonometric polynomial $p \in \mathcal{P}(\mathbb{T}) = \mathcal{P}_\pm(\mathbb{T})$ is nonnegative on \mathbb{T} , then by the Féjer-Riesz theorem there exists an analytic trigonometric polynomial $q \in \mathcal{P}_+(\mathbb{T})$ such that $p(z) = |q(z)|^2$ for all $z \in \mathbb{T}$.

9. Approximation. In this section we intend to apply our approach to approximating nonnegative polynomials by sums of squares of moduli of rational functions. The method presented here could be a fertile source of other approximation results of this kind.

Let T be a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$. Denote by $\Sigma_T^2(\mathbb{C})$ the set of all polynomials $q \in \mathbb{C}_T[z, \bar{z}]$ for which there exist finitely many rational functions $q_1, \dots, q_k \in \mathbb{C}_{\mathfrak{N}_+}(z, \bar{z})$ such that $q(z, \bar{z}) = \sum_{j=1}^k |q_j(z, \bar{z})|^2$ for all $z \in \mathbb{C}_*$. Let $\mathcal{P}_T^+(\mathbb{C})$ stand for the set of all polynomials $q \in \mathbb{C}_T[z, \bar{z}]$ such that $q(z, \bar{z}) \geq 0$ for all $z \in \mathbb{C}$. We shall regard $\Sigma_T^2(\mathbb{C})$, $\mathcal{P}_T^+(\mathbb{C})$ and $\mathbb{C}_T[z, \bar{z}]$ as sets of complex functions on \mathbb{C} . Given $p \in \mathbb{C}[z, \bar{z}]$ of the form $p(z, \bar{z}) = \sum_{m,n \geq 0} a_{m,n} z^m \bar{z}^n$, we set $\|p\|_{\text{co}} = \max\{|a_{m,n}| : m, n \geq 0\}$. The function $\|\cdot\|_{\text{co}}$ is a norm on $\mathbb{C}[z, \bar{z}]$. Recall that the finest locally convex topology on a linear space V is given by the family of all seminorms on V .

PROPOSITION 36. *Let T be a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$ such that $(n, m) \in T$ for all $(m, n) \in T$, and $(n, n) \in T$ for all $n \in \mathbb{Z}_+$. Then the set $\Sigma_T^2(\mathbb{C})$ is dense in $\mathcal{P}_T^+(\mathbb{C})$ with respect to the finest locally convex topology τ on $\mathbb{C}_T[z, \bar{z}]$. In particular, this is the case for the topology of uniform convergence on compact subsets of \mathbb{C} and the topology given by the norm $\|\cdot\|_{\text{co}}$.*

PROOF. It only suffices to discuss the case of the finest locally convex topology. We regard $\mathbb{C}[z, \bar{z}]$ as a $*$ -algebra of complex functions on \mathbb{C} with involution $q^*(z, \bar{z}) = \overline{q(z, \bar{z})}$ for $z \in \mathbb{C}$. Since the set T is assumed to have a symmetry property, we see that $\mathbb{C}_T[z, \bar{z}]$ is a vector subspace of $\mathbb{C}[z, \bar{z}]$ such that $q^* \in \mathbb{C}_T[z, \bar{z}]$ for every $q \in \mathbb{C}_T[z, \bar{z}]$. It is then clear that

$$\Sigma_T^2(\mathbb{C}) \subset \mathcal{P}_T^+(\mathbb{C}) \subset \mathbb{C}_T[z, \bar{z}]_{\mathfrak{h}} \stackrel{\text{def}}{=} \{q \in \mathbb{C}_T[z, \bar{z}] : q = q^*\}.$$

Suppose that, contrary to our claim, there exists $q_0 \in \mathcal{P}_T^+(\mathbb{C})$ which does not belong to the τ -closure of $\Sigma_T^2(\mathbb{C})$. By the separation theorem (cf. [46, Theorem 3.4 (b)]), there exist a real-linear functional $\tilde{A} : \mathbb{C}_T[z, \bar{z}] \rightarrow \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that

$$(42) \quad \tilde{A}(q_0) < \gamma \leq \tilde{A}(q), \quad q \in \Sigma_T^2(\mathbb{C}).$$

Since $\Sigma_T^2(\mathbb{C})$ has the property that $tq \in \Sigma_T^2(\mathbb{C})$ for all $q \in \Sigma_T^2(\mathbb{C})$ and $t \in [0, \infty)$, we deduce from (42) that

$$\tilde{A}(q_0) < 0 \leq \tilde{A}(q), \quad q \in \Sigma_T^2(\mathbb{C}),$$

Define $\Lambda : \mathbb{C}_T[z, \bar{z}] \rightarrow \mathbb{C}$ by

$$\Lambda(q) = \tilde{A}(\Re q) + i \tilde{A}(\Im q), \quad q \in \mathbb{C}_T[z, \bar{z}],$$

with $\Re q \stackrel{\text{def}}{=} (q + q^*)/2$ and $\Im q \stackrel{\text{def}}{=} (q - q^*)/(2i)$. It is easily seen that Λ is a complex-linear functional such that $\Lambda(q) = \tilde{A}(q)$ for all $q \in \mathbb{C}_T[z, \bar{z}]_{\mathfrak{h}}$. Hence, in

view of the inclusion $\{q_0\} \cup \Sigma_T^2(\mathbb{C}) \subset \mathbb{C}_T[z, \bar{z}]_{\mathfrak{h}}$, we have

$$(43) \quad \Lambda(q_0) < 0 \leq \Lambda(q), \quad q \in \Sigma_T^2(\mathbb{C}).$$

Since Λ is of the form

$$\Lambda(q) = \sum_{(m,n) \in T} p_{m,n} c_{m,n} \quad \text{for } q \in \mathbb{C}_T[z, \bar{z}]: q(z, \bar{z}) = \sum_{(m,n) \in T} p_{m,n} z^m \bar{z}^n,$$

where $\{c_{m,n}\}_{(m,n) \in T}$ is a system of complex numbers uniquely determined by Λ , we deduce from the weak inequality in (43) and implication (v) \Rightarrow (iv) of Theorem 20 that $\Lambda(q_0) \geq 0$, which contradicts the strict inequality in (43). \square

Consider now the case of $T = \mathfrak{N} = \mathbb{Z}_+ \times \mathbb{Z}_+$. Observe that $\widehat{\Sigma}_{\mathfrak{N}}^2(\mathbb{C})$, the convex cone of all polynomials $q \in \mathbb{C}[z, \bar{z}]$ for which there exist finitely many polynomials $p_1, \dots, p_n \in \mathbb{C}[z, \bar{z}]$ such that $q(z, \bar{z}) = \sum_{j=1}^n |p_j(z, \bar{z})|^2$ for all $z \in \mathbb{C}$, is closed with respect to the finest locally convex topology of $\mathbb{C}[z, \bar{z}]$ (see the proof of [6, Theorem 6.3.5]). Evidently, the convex cone $\mathcal{P}_{\mathfrak{N}}^+(\mathbb{C})$ is closed with respect to the same topology. It is clear that

$$(44) \quad \widehat{\Sigma}_{\mathfrak{N}}^2(\mathbb{C}) \subset \Sigma_{\mathfrak{N}}^2(\mathbb{C}) \subset \mathcal{P}_{\mathfrak{N}}^+(\mathbb{C}).$$

Owing to the proof of [6, Theorem 6.3.5], $\widehat{\Sigma}_{\mathfrak{N}}^2(\mathbb{C})$ is a proper subset of $\mathcal{P}_{\mathfrak{N}}^+(\mathbb{C})$. Hence, by Proposition 36, the first inclusion in (44) is proper as well. The open question whether the second inclusion in (44) is proper has already been posed in the paragraph preceding Proposition 24.

COROLLARY 37. *Let*

$$q(z, \bar{z}) = \sum_{m,n \geq 0} a_{m,n} z^m \bar{z}^n, \quad z \in \mathbb{C},$$

be a nonnegative polynomial with coefficients in \mathbb{C} . Then for every $\varepsilon > 0$, there exists a nonnegative polynomial $q_\varepsilon \in \Sigma_{\mathfrak{N}}^2(\mathbb{C})$ such that

$$\begin{aligned} q_\varepsilon(z, \bar{z}) &= \sum_{m,n \geq 0} a_{m,n}^{(\varepsilon)} z^m \bar{z}^n, \quad z \in \mathbb{C} \quad (a_{m,n}^{(\varepsilon)} \in \mathbb{C}), \\ |a_{m,n} - a_{m,n}^{(\varepsilon)}| &\leq \varepsilon |a_{m,n}|, \quad m, n \in \mathbb{Z}_+, m \neq n, \\ |a_{m,n} - a_{m,n}^{(\varepsilon)}| &\leq \varepsilon, \quad m, n \in \mathbb{Z}_+. \end{aligned}$$

PROOF. It is enough to consider the case of $q \neq 0$. Since the polynomial q is nonnegative, we deduce that the set

$$T = \{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : a_{m,n} \neq 0\} \cup \{(n, n) : n \in \mathbb{Z}_+\}$$

satisfies the assumptions of Proposition 36, and $q \in \mathcal{P}_T^+(\mathbb{C})$. By this proposition, there exists $q_\varepsilon \in \Sigma_T^2(\mathbb{C})$ such that $\|q - q_\varepsilon\|_{\text{co}} \leq \tilde{\varepsilon}$, where

$$\tilde{\varepsilon} = \varepsilon \cdot \min \left\{ 1, \min \left\{ |a_{m,n}| : m, n \geq 0, a_{m,n} \neq 0 \right\} \right\} > 0.$$

This completes the proof. \square

We now discuss the case of trigonometric polynomials (see Section 7 for notation). Let T be a subset of \mathbb{Z}^\times and let $\Sigma_T^2(\mathbb{T}^\times)$ stand for the set of all trigonometric polynomials $q \in \mathcal{P}_T(\mathbb{T}^\times)$ for which there exist finitely many analytic trigonometric polynomials $q_1, \dots, q_k \in \mathcal{P}_+(\mathbb{T}^\times)$ such that $q(z) = \sum_{j=1}^k |q_j(z)|^2$ for all $z \in \mathbb{T}^\times$. Denote by $\mathcal{P}_T^+(\mathbb{T}^\times)$ the set of all trigonometric polynomials $q \in \mathcal{P}_T(\mathbb{T}^\times)$

such that $q(z) \geq 0$ for all $z \in \mathbb{T}^\mathbb{N}$. Given $p \in \mathcal{P}(\mathbb{T}^\mathbb{N})$ of the form (35), we set $\|p\|'_{\text{co}} = \max\{|a_\alpha| : \alpha \in \mathbb{Z}^\mathbb{N}\}$. It is clear that $\|\cdot\|'_{\text{co}}$ is a norm on $\mathcal{P}(\mathbb{T}^\mathbb{N})$.

Arguing as in the proof of Proposition 36 (using Theorem 34 instead of Theorem 20), we get another approximation result.

PROPOSITION 38. *Let T be a subset of $\mathbb{Z}^\mathbb{N}$ such that $0 \in T = -T$. Then the set $\Sigma_T^2(\mathbb{T}^\mathbb{N})$ is dense in $\mathcal{P}_T^+(\mathbb{T}^\mathbb{N})$ with respect to the finest locally convex topology on $\mathcal{P}_T(\mathbb{T}^\mathbb{N})$. In particular, this is the case for the topology of uniform convergence on $\mathbb{T}^\mathbb{N}$ and the topology given by the norm $\|\cdot\|'_{\text{co}}$.*

COROLLARY 39. *Let*

$$q(z) = \sum_{\alpha \in \mathbb{Z}^\mathbb{N}} a_\alpha z^\alpha, \quad z \in \mathbb{T}^\mathbb{N},$$

be a nonnegative trigonometric polynomial with coefficients in \mathbb{C} . Then for every $\varepsilon > 0$, there exists a nonnegative trigonometric polynomial $q_\varepsilon \in \Sigma_{\mathbb{Z}^\mathbb{N}}^2(\mathbb{T}^\mathbb{N})$ such that

$$q_\varepsilon(z) = \sum_{\alpha \in \mathbb{Z}^\mathbb{N}} a_\alpha^{(\varepsilon)} z^\alpha, \quad z \in \mathbb{T}^\mathbb{N} \quad (a_\alpha^{(\varepsilon)} \in \mathbb{C}),$$

$$|a_\alpha - a_\alpha^{(\varepsilon)}| \leq \varepsilon |a_\alpha|, \quad \alpha \in \mathbb{Z}^\mathbb{N} \setminus \{(0, \dots, 0)\},$$

and

$$(45) \quad |a_\alpha - a_\alpha^{(\varepsilon)}| \leq \varepsilon, \quad \alpha \in \mathbb{Z}^\mathbb{N}.$$

PROOF. It is enough to consider the case of $q \neq 0$. Since the trigonometric polynomial q is nonnegative, we see that the set $T = \{\alpha \in \mathbb{Z}^\mathbb{N} : a_\alpha \neq 0\} \cup \{(0, \dots, 0)\}$ satisfies the assumptions of Proposition 38, and $q \in \mathcal{P}_T^+(\mathbb{T}^\mathbb{N})$. By this proposition, there exists $q_\varepsilon \in \Sigma_T^2(\mathbb{T}^\mathbb{N})$ such that $\|q - q_\varepsilon\|'_{\text{co}} \leq \tilde{\varepsilon}$, where

$$\tilde{\varepsilon} = \varepsilon \cdot \min \left\{ 1, \min \left\{ |a_\alpha| : \alpha \in \mathbb{Z}^\mathbb{N}, a_\alpha \neq 0 \right\} \right\} > 0.$$

This completes the proof. \square

For the case of $T = \mathbb{Z}^\mathbb{N}$ it is well known that $\mathcal{P}_{\mathbb{Z}^\mathbb{N}}^+(\mathbb{T}^\mathbb{N}) \setminus \Sigma_{\mathbb{Z}^\mathbb{N}}^2(\mathbb{T}^\mathbb{N}) \neq \emptyset$ if $\mathbb{N} \geq 2$ (cf. [47, page 51]). Recently Dritschel proved that each (strictly) positive trigonometric polynomial in $\mathcal{P}(\mathbb{T}^\mathbb{N})$ belongs to $\Sigma_{\mathbb{Z}^\mathbb{N}}^2(\mathbb{T}^\mathbb{N})$ (see [22, Corollary 5.2] where operator valued trigonometric polynomials are considered; see also [29] for related questions concerning scalar trigonometric polynomials in two variables). The proof of Dritschel's result is based on his Theorem 5.1 which in the scalar case coincides with the major part of our Corollary 39 (except for (45)).

10. The truncated two-sided complex moment problem. Let \mathfrak{Z} stand for the $*$ -semigroup $(\mathbb{Z} \times \mathbb{Z}, +, *)$ with coordinatewise defined addition as semigroup operation, and involution $(m, n)^* = (n, m)$. Given a subset T of \mathfrak{Z} , we denote by $\Sigma_T^2(\mathbb{C}_*)$ the set of all rational functions $q \in \mathbb{C}_T(z, \bar{z})$ for which there exist finitely many rational functions $q_1, \dots, q_k \in \mathbb{C}_3(z, \bar{z})$ such that $q(z, \bar{z}) = \sum_{j=1}^k |q_j(z, \bar{z})|^2$ for all $z \in \mathbb{C}_*$. Let $\mathcal{P}_T^+(\mathbb{C}_*)$ stand for the set of all rational functions $q \in \mathbb{C}_T(z, \bar{z})$ such that $q(z, \bar{z}) \geq 0$ for all $z \in \mathbb{C}_*$. We regard $\Sigma_T^2(\mathbb{C}_*)$, $\mathcal{P}_T^+(\mathbb{C}_*)$ and $\mathbb{C}_T(z, \bar{z})$ as sets of complex functions on \mathbb{C}_* . Given $p \in \mathbb{C}_3(z, \bar{z})$ of the form $p(z, \bar{z}) = \sum_{m, n \in \mathbb{Z}} a_{m, n} z^m \bar{z}^n$, we set $\|p\|_{\text{co}} = \max\{|a_{m, n}| : m, n \in \mathbb{Z}\}$. The function $\|\cdot\|_{\text{co}}$ is a norm on $\mathbb{C}_3(z, \bar{z})$.

It was proved by Bisgaard (cf. [7]) that the $*$ -semigroup \mathfrak{Z} is semiperfect (in fact, as noted in [55, Theorem 24], \mathfrak{Z} is operator semiperfect). This result enables us to apply our machinery. Below, we formulate counterparts of Theorem 20 (a shorter version), Proposition 36 and Corollary 37. Their proofs are analogical.

THEOREM 40. *Let T be a symmetric subset of \mathfrak{Z} (i.e. $(n, m) \in T$ for all $(m, n) \in T$) containing the diagonal¹¹ $\{(n, n) : n \in \mathbb{Z}\}$, and let $Z \subset \mathbb{C}_*$ be a determining set for $\mathbb{C}[z, \bar{z}]$. Then for any system of complex numbers $\{c_{m,n}\}_{(m,n) \in T}$, the following conditions are equivalent:*

(i) *there exists a positive Borel measure μ on \mathbb{C}_* such that*

$$c_{m,n} = \int_{\mathbb{C}_*} z^m \bar{z}^n \mu(dz), \quad (m, n) \in T,$$

(ii) *$\sum_{(m,n) \in T} p_{m,n} c_{m,n} \geq 0$ for every finite system $\{p_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ such that $\sum_{(m,n) \in T} p_{m,n} z^m \bar{z}^n \geq 0$ for all $z \in \mathbb{C}_*$,*

(iii) *$\sum_{(m,n) \in T} p_{m,n} c_{m,n} \geq 0$ for every finite system $\{p_{m,n}\}_{(m,n) \in T} \subset \mathbb{C}$ for which there exist finitely many rational functions $q_1, \dots, q_k \in \mathbb{C}_3(z, \bar{z})$ such that $\sum_{(m,n) \in T} p_{m,n} z^m \bar{z}^n = \sum_{j=1}^k |q_j(z, \bar{z})|^2$ for all $z \in Z$.*

PROPOSITION 41. *Let T be as in Theorem 40. Then the set $\Sigma_T^2(\mathbb{C}_*)$ is dense in $\mathcal{P}_T^+(\mathbb{C}_*)$ with respect to the finest locally convex topology on $\mathbb{C}_T(z, \bar{z})$. In particular, this is the case for the topology of uniform convergence on compact subsets of \mathbb{C}_* and the topology given by the norm $\|\cdot\|_{\text{co}}$.*

COROLLARY 42. *Let*

$$q(z, \bar{z}) = \sum_{m,n \in \mathbb{Z}} a_{m,n} z^m \bar{z}^n, \quad z \in \mathbb{C}_*,$$

be a nonnegative rational function with coefficients in \mathbb{C} . Then for every $\varepsilon > 0$, there exists a nonnegative rational function $q_\varepsilon \in \Sigma_3^2(\mathbb{C}_)$ such that*

$$q_\varepsilon(z, \bar{z}) = \sum_{m,n \in \mathbb{Z}} a_{m,n}^{(\varepsilon)} z^m \bar{z}^n, \quad z \in \mathbb{C}_* \quad (a_{m,n}^{(\varepsilon)} \in \mathbb{C}),$$

$$|a_{m,n} - a_{m,n}^{(\varepsilon)}| \leq \varepsilon |a_{m,n}|, \quad m, n \in \mathbb{Z}, m \neq n,$$

$$|a_{m,n} - a_{m,n}^{(\varepsilon)}| \leq \varepsilon, \quad m, n \in \mathbb{Z}.$$

Yet another proof of Proposition 41 consists in deriving it from Proposition 36. In turn, Corollary 42 can be deduced directly from Proposition 41. However, Theorem 40 does not seem to be an easy consequence of Theorem 20.

11. Concluding remarks. A result of Bisgaard (cf. [7, 10]) states that the Cartesian product of two $*$ -semigroups one of which is (operator) perfect and the other (operator) semiperfect is (operator) semiperfect. This fact can be applied to $*$ -semigroups \mathbb{Z}^\times (operator perfect) and \mathfrak{N}_+ (operator semiperfect). Employing our method in this particular context we obtain appropriate results for mixed polynomials

$$\sum_{\alpha \in \mathbb{Z}^\times} \sum_{m,n \geq 0} a_{\alpha,m,n} w^\alpha z^m \bar{z}^n, \quad w \in \mathbb{T}^\times, z \in \mathbb{C}.$$

¹¹ Again, as noticed in footnote 10, this is the form condition (14) takes now.

Fortunately, there is a variety of $*$ -semigroups which are semiperfect or operator semiperfect, and for which we can find convenient description of their dual $*$ -semigroups and Laplace transforms (see e.g. [6, 13, 40, 51, 8, 9, 41, 24, 57]).

As we now show, a finitely generated $*$ -semigroup with unit can be modelled as a quotient of the $*$ -semigroup $\mathbb{Z}_+^\varkappa \times \mathbb{Z}_+^\varkappa$, which gives rise to yet another way of describing its dual. Suppose that a commutative $*$ -semigroup \mathfrak{S} with unit is finitely generated. Then there exists a finite system $\mathbf{e} = (e_1, \dots, e_\varkappa)$ of elements of \mathfrak{S} such that the mapping

$$\Phi = \Phi_{\mathbf{e}}: \mathbb{Z}_+^\varkappa \times \mathbb{Z}_+^\varkappa \ni (\alpha, \beta) \mapsto \mathbf{e}^\alpha \mathbf{e}^{\beta*} \in \mathfrak{S}$$

is a surjection, where $\mathbf{e}^\alpha = e_1^{\alpha_1} \dots e_\varkappa^{\alpha_\varkappa}$ and $\mathbf{e}^{\beta*} = e_1^{*\beta_1} \dots e_\varkappa^{*\beta_\varkappa}$ with $\alpha = (\alpha_1, \dots, \alpha_\varkappa)$ and $\beta = (\beta_1, \dots, \beta_\varkappa)$. Let us regard $\mathfrak{N}_\varkappa \stackrel{\text{def}}{=} \mathbb{Z}_+^\varkappa \times \mathbb{Z}_+^\varkappa$ as a $*$ -semigroup with coordinatewise defined addition as semigroup operation and involution $(\alpha, \beta)^* = (\beta, \alpha)$ for $\alpha, \beta \in \mathbb{Z}_+^\varkappa$. Then Φ is a $*$ -epimorphism of $*$ -semigroups, and consequently the relation $R = R_\Phi$ on \mathfrak{N}_\varkappa defined by

$$(\alpha, \beta) R (\alpha', \beta') \quad \text{if} \quad \Phi(\alpha, \beta) = \Phi(\alpha', \beta')$$

is a congruence. As a consequence, the $*$ -semigroup \mathfrak{S} is $*$ -isomorphic to the quotient $*$ -semigroup \mathfrak{N}_\varkappa/R (consult [34, Theorem I.1.5]). This means that the $*$ -semigroups \mathfrak{N}_\varkappa/R , $\varkappa \geq 1$, are model $*$ -semigroups in the category of finitely generated $*$ -semigroups. We now describe the dual $*$ -semigroup of the model $*$ -semigroup \mathfrak{N}_\varkappa/R . In what follows, we regard \mathbb{C}^\varkappa as a $*$ -semigroup with coordinatewise defined multiplication as semigroup operation and involution $(z_1, \dots, z_\varkappa)^* = (\bar{z}_1, \dots, \bar{z}_\varkappa)$. First, note that the set

$$\mathfrak{F}_R = \{z \in \mathbb{C}^\varkappa: z^\alpha \bar{z}^\beta = z^{\alpha'} \bar{z}^{\beta'} \text{ whenever } (\alpha, \beta) R (\alpha', \beta')\}$$

is a $*$ -subsemigroup of \mathbb{C}^\varkappa . It is a matter of routine to verify that the mapping

$$(46) \quad \Psi: \mathfrak{F}_R \rightarrow \mathfrak{X}_{\mathfrak{N}_\varkappa/R}, \quad \Psi(z)[(\alpha, \beta)]_R = z^\alpha \bar{z}^\beta \text{ for } (\alpha, \beta) \in \mathfrak{N}_\varkappa, z \in \mathfrak{F}_R,$$

where $[(\alpha, \beta)]_R$ is the equivalence class of (α, β) with respect to the relation R , is a $*$ -isomorphism of $*$ -semigroups (hint: the mapping $\mathbb{C}^\varkappa \ni z \mapsto \varphi_z \in \mathfrak{X}_{\mathfrak{N}_\varkappa}$, where $\varphi_z(\alpha, \beta) = z^\alpha \bar{z}^\beta$ for $(\alpha, \beta) \in \mathfrak{N}_\varkappa$, is a $*$ -isomorphism of $*$ -semigroups). Applying the measure transport theorem to (46), one can rewrite the integrals $\int_{\mathfrak{X}_{\mathfrak{S}}} \hat{s} d\mu$ that appear in the conditions (ii) of Theorems 14 and 15 in the form $\int_{\mathfrak{F}_R} z^\alpha \bar{z}^\beta d\tilde{\mu}(z)$ which resembles the solution of the multidimensional complex moment problem. However, the example of the $*$ -semigroup \mathfrak{N}_+ in Section 5 shows that this general approach is not always efficient.

APPENDIX

First, we recall the Riesz-Haviland theorem which completely characterizes multidimensional real moment sequences (cf. [30, 31]; see also [44] for the earlier solution of the one-dimensional real moment problem). Below $L_{\mathbf{a}}$ is a unique linear functional defined on the linear space of all complex polynomials in \varkappa indeterminates given by $L_{\mathbf{a}}(\mathbf{x}^{\mathbf{n}}) = a_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}_+^\varkappa$ (with standard multiindex notation). Likewise, $L_{\mathbf{c}}$ appearing in Theorem B is determined by $L_{\mathbf{c}}(z^{\mathbf{m}} \bar{z}^{\mathbf{n}}) = c_{\mathbf{m}, \mathbf{n}}$, $(\mathbf{m}, \mathbf{n}) \in \mathbb{Z}_+^{2\varkappa}$.

THEOREM A. *If $\mathbf{a} = \{a_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}_+^\varkappa}$ is a sequence of real numbers and Z is a closed subset of \mathbb{R}^\varkappa , then the following conditions are equivalent:*

(a) *there exists a positive Borel measure μ on \mathbb{R}^\varkappa such that*

$$a_{\mathbf{n}} = \int_Z \mathbf{x}^{\mathbf{n}} d\mu(\mathbf{x}), \quad \mathbf{n} \in \mathbb{Z}_+^\varkappa,$$

(b) *$L_{\mathbf{a}}(p) \geq 0$ whenever p is a complex polynomial in \varkappa indeterminates such that $p(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in Z$.*

Next, we formulate the complex variant of the Riesz-Haviland theorem. It can be deduced from Theorem A (with $\varkappa = 2$) in an elementary way as indicated below.

THEOREM B. *If $\mathbf{c} = \{c_{m,n}\}_{m,n=0}^\infty$ is a sequence of complex numbers and Z is a closed subset of \mathbb{C} , then the following conditions are equivalent:*

(a) *there exists a positive Borel measure μ on \mathbb{C} such that*

$$c_{m,n} = \int_Z z^m \bar{z}^n d\mu(z), \quad m, n \geq 0,$$

(b) *$L_{\mathbf{c}}(p) \geq 0$ whenever p is a complex polynomial in z and \bar{z} such that $p(z, \bar{z}) \geq 0$ for all $z \in Z$.*

Now we relate the two-dimensional real moment problem to the one-dimensional complex moment problem. It will be done much in the spirit of [55, Appendix], though by more elementary means. The same proof works in the multidimensional case.

For every $(m, n) \in \mathbb{Z}_+^2$, there exist finite systems $\{\alpha_{k,l}^{m,n}\}_{k,l=0}^\infty$ and $\{\beta_{k,l}^{m,n}\}_{k,l=0}^\infty$ of complex numbers uniquely determined by the following identities

$$(47) \quad (x + iy)^m (x - iy)^n = \sum_{k,l \geq 0} \alpha_{k,l}^{m,n} x^k y^l, \quad x, y \in \mathbb{R},$$

$$(48) \quad \left(\frac{z + \bar{z}}{2}\right)^m \left(\frac{z - \bar{z}}{2i}\right)^n = \sum_{k,l \geq 0} \beta_{k,l}^{m,n} z^k \bar{z}^l, \quad z \in \mathbb{C}.$$

It can be readily checked that

$$(49) \quad \sum_{i,j \geq 0} \alpha_{i,j}^{m,n} \beta_{k,l}^{i,j} = \sum_{i,j \geq 0} \beta_{i,j}^{m,n} \alpha_{k,l}^{i,j} = \delta_{m,k} \delta_{n,l}, \quad k, l, m, n \in \mathbb{Z}_+.$$

Given a sequence $\mathbf{a} = \{a_{m,n}\}_{m,n=0}^\infty \subset \mathbb{C}$, we define $\mathbf{c}(\mathbf{a}) = \{c_{m,n}(\mathbf{a})\}_{m,n=0}^\infty$ via

$$c_{m,n}(\mathbf{a}) = \sum_{k,l \geq 0} \alpha_{k,l}^{m,n} a_{k,l}.$$

By (49) the mapping $\mathbb{C}^{\mathbb{Z}_+ \times \mathbb{Z}_+} \ni \mathbf{a} \mapsto \mathbf{c}(\mathbf{a}) \in \mathbb{C}^{\mathbb{Z}_+ \times \mathbb{Z}_+}$ is a linear isomorphism, and

$$(50) \quad a_{m,n} = \sum_{k,l \geq 0} \beta_{k,l}^{m,n} c_{k,l}(\mathbf{a}), \quad m, n \in \mathbb{Z}_+,$$

for all $\mathbf{a} = \{a_{m,n}\}_{m,n=0}^\infty \subset \mathbb{C}$. Below we identify \mathbb{C} with \mathbb{R}^2 .

PROPOSITION 43. *Let Z be a subset of \mathbb{C} . Then a sequence $\mathbf{a} = \{a_{m,n}\}_{m,n=0}^\infty \subset \mathbb{R}$ satisfies the condition (a) (resp. (b)) of Theorem A with $\varkappa = 2$ if and only if the sequence $\mathbf{c}(\mathbf{a})$ satisfies the condition (a) (resp. (b)) of Theorem B.*

PROOF. If $a_{m,n} = \int_Z x^m y^n d\mu(x, y)$ for all $m, n \in \mathbb{Z}_+$, then

$$c_{m,n}(\mathbf{a}) = \sum_{k,l \geq 0} \alpha_{k,l}^{m,n} a_{k,l} = \int_Z \sum_{k,l \geq 0} \alpha_{k,l}^{m,n} x^k y^l d\mu(x, y) \stackrel{(47)}{=} \int_Z z^m \bar{z}^n d\mu(z).$$

Conversely, if $\mathbf{c}(\mathbf{a})$ is a complex moment sequence with a representing measure μ on Z , then

$$a_{m,n} \stackrel{(50)}{=} \sum_{k,l \geq 0} \beta_{k,l}^{m,n} c_{k,l}(\mathbf{a}) = \int_Z \sum_{k,l \geq 0} \beta_{k,l}^{m,n} z^k \bar{z}^l d\mu(z) \stackrel{(48)}{=} \int_Z x^m y^n d\mu(x, y),$$

which completes the proof of the equivalence of both conditions (a).

Similar calculation based on (47), (48) and (50) justifies the equivalence of the conditions (b). \square

Since the positive functional $L_{\mathbf{a}}$ given by a complex sequence $\mathbf{a} = \{a_{m,n}\}_{m,n=0}^{\infty}$ is automatically Hermitian (i.e. $L_{\mathbf{a}}(\bar{p}) = \overline{L_{\mathbf{a}}(p)}$), the sequence \mathbf{a} is necessarily real. This allows us to deduce Theorem B from Theorem A and Proposition 43.

Acknowledgments. The very early version of this paper was designated for Kreĭn's anniversary volume. However, due to its growing capacity we have been exceeding all consecutive deadlines; let us thank, by the way, Professor Vadim Adamyan for his patience in negotiating them.

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